Ask Yourself 3: The Return of the Quizzing

The Twelvefold Way:
$|\{f : k \to n\}|$
How many ways to sort k balls into n boxes?

	Arbitrary any sorting	Injective max 1 ball per box	Surjective each box gets ball
Distinct Balls Distinct Boxes	n^k	$\frac{n!}{(n-k)!}$	$n!\left\{ {k \atop n} \right\}$
Identical Balls Distinct Boxes	$\binom{n+k-1}{k}$	$\binom{n}{k}$	$\binom{k-1}{n-1}$
Distinct Balls Identical Boxes	$\sum_{j=0}^{n} \left\{ {k \atop j} \right\}$	1 if $k \leq n$	$\left\{ {k \atop n} \right\}$
Identical Balls Identical Boxes	$p_{\leq n}(k)$	1 if $k \leq n$	$p_n(k)$

1. Could I explain and give a way to compute every entry in the 12fold way?

2. Could I give a definition for any of these jargon words in context?

   function, injection, surjection, permutation, inverse, generating function of a sequence, partition, integer partitions, permutation/symmetry group, symmetry/automorphism group of a graph, equivalence relation, permutation group action on functions, equivalence modulo a group, equivalence class/orbit, stabelizer, fixed set, Polya cycle index, recurrence relation, linear recurrence relation, solution set to a linear recurrence relation

3. Could I state or explain the following major results?

   (i) The Binomial Theorem (but as a generating function!)

   (ii) The Generating Function of Integer Partitions

   (iii) Burnside's Lemma

   (iv) Polya's Enumeration of Functions Mod G

   (v) The Solutions of Linear Recurrence Relation

4. Could I solve linear recurrences?

   (a) Find a closed formula in n for sequence a_n with $a_0 = 1$, $a_1 = 4$, and $a_2 = 11$ and which satisfies the the recurrence

   $$a_{n+3} = 5a_{n+2} - 7a_{n+1} + 3a_n.$$

   (edit: corrected $-3 \to 3$) and find closed form for the generating function.

   (b) Solve the recurrence equation $r_{n+2} = r_{n+1} + 2r_n$ with $r_0 = 1$ and $r_2 = 3$.

(c) Find the solution set of all possible sequences satisfying the recurrence. (You could factor the polynomial using the rational root theorem. Or a computer.)

$$a_{n+6} = 12a_{n+5} - 58a_{n+4} + 144a_{n+3} - 193a_{n+2} + 132a_{n+1} - 36a_n$$

5. Can I read permutation cycle notation and do I know what it means? Could I compute compositions? Remembering that we apply them as functions from the right? What is the Polya cycle index for each permutation?

   (a) (157)(1957)

   (b) (12345)(15432)

   (c) (12)(25)(53)(31)(32)(15)

6. Could I use generating functions and Polya counting to solve counting questions? Can I find the closed forms of generating functions? J. R. R. Tolkien's *The Lord of the Rings* is a trilogy of fantasy novels that come with 2 prequels, *The Hobbit*, and the *The Silmarillion*.

   (a) Amazon just bought the rights from the Tolkien estate for \$250 million [1]. That's just the rights. Not even production costs. Find a generating function for how many ways can an \$ n budget can be distributed among the 5 books.

   (b) The trilogy itself has 455,125 words, with each book over 100,000 words. Use a generating function to find the number of possible ways to distribute the words among the trilogy, so that each book has over 100,000 words.

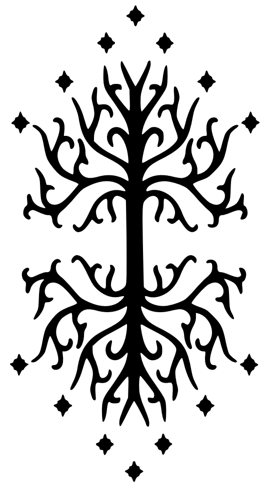
   (c) Construct a generating function to answer: How many distinct collections of n eponymous *rings of power* are there, given that there are 3 rings for the elven-kings under the sky, 7 for the dwarf-lords in their halls of stone, 9 for mortal men doomed to die, and 1 for the Dark Lord on his dark throne.

   (d) The golden coins of Dale that Bilbo brought back from his adventure to steal a dragon's treasure come in rather large quantities compared to the currency of his home, the Shire. There are three types of coins are worth \$11, \$21, and \$31 Shire dollars. Gandalf (a wizard) mentions that Bilbo could make any number of Shire dollars higher than \$200, and Bilbo believes him (he's a wizard). How could Bilbo use a generating function to find all the different values of Shire dollars he **cannot** make out of Dale coins?

   (e) Legolas, a very good elven archer, like to compete with his friend Gimili in orc killing contests. Legolas never ever misses, but occasionally particularly tough opponents will block or survive an arrow, and sometimes he kills two orcs with one arrow. If we watch the films and write down the sequence of arrows fired, and how many enemies they kill (limit 0, 1, or 2). Use two variables to make a generating function that tracks number of different strings we might write with n shots and $m \leq n$ kills.

[1] This review sheet is fair use, but don't tell the Tolkien estate, thanks.

(f) As Galadriel, the Lady of Lothlorien, peers into her mirror of future sight she beholds the following symbol, doubtless a sign of the seven and seven Valar, and of victory for Gondor.



Represent the symmetries of the symbol as permutations of the 14 stars. (There are 4, given by mirror reflections) How many distinct ways (up to symmetry), might the stars be decorated with pearls, diamonds, and sapphires? Give the Polya cycle index of the group.

(g) Supposing Sauron obtained all 20 rings, he couldn't wear them since SPOILER ALERT he is a lidless eye wreathed in flame and has no fingers. If he instead put them on a crown (as Morgoth did the Samirils) with an 5-order rotation as shown in Figure 1. How many ways (modulo just rotation, forget the reflecting) can 20

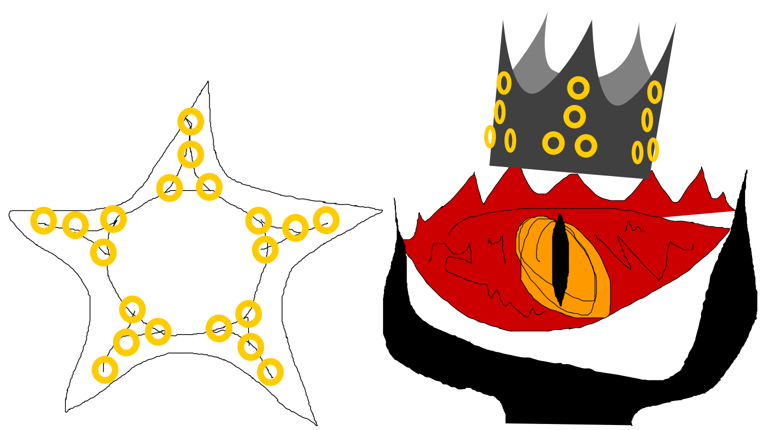


Figure 1: (Left) 20 rings arranged in an order 5 symmetric crown (Right) The Lidless Eye, Crowned Lord of All Rings

spots in the crown be filled with 4 types of rings? Find the Polya Cycle Index of the group 5 of rotation permutations of rings on the crown. Use a variable substitution to find the generating function that tracks the number of crowns by the number of each type of ring. How many crowns are there with 1,3,7,and 9?