Solving Linear Recurrence Relations with Linear Algebra:
A 3012 Addendum Shane Scott

This note uses a lot of linear algebra, and assumes you know some of the terms and how to multiply matrices. You might want to have the following review sheet handy for recalling some of the jargon as you read:

Ivan Savov's Very Good, Blisteringly Brief Guide to Linear Algebra

1 Linear Recurrence

Recurrence relations give a great way to count inductively. If you know the first few numbers in a sequence, the recurrence relation tells you how to compute the rest of the sequence.

E.g. The famous Fibonacci sequence $0, 1, 1, 2, 3, 5, 8, 13, \ldots$ satisfies the recurrence relation $f_{n+2} = f_{n+1} + f_n$, so that each number is the sum of the previous two numbers in the sequence.

E.g. If s_n is the number of ternary strings length n that don't contain 102 as a substring then:

$$s_{n+1} = 3s_n - s_{n-2}$$

(Excercise: why?) Then to find all s_n we would need to figure out the first 3 terms $s_0 = 1$, $s_1 = 3$, $s_2 = 9$ but then the rest can be computed from the recurrence relation!

$$1, 3, 9, 26, 75, 216, 622, \ldots$$

Remark If I want a particular term of a recurrence sequence, say the 100th Fibonacci number, it's pretty inefficient to have to compute all the numbers that come before it. It would be convenient to have a *closed form formula* for f_n just as a function of the number n, rather than as recursive formula. That would also tell me more about the sequence, like its asymptotics $\mathcal{O}$, or allow me to analyze it without necessarily knowing the first few terms.

Question The Fibonacci numbers satisfy the recurrence $f_{n+2} = f_{n+1} + f_n$, but they aren't the only ones! So do the Lucas numbers:

$$1, 3, 4, 7, 11, 18, 29, 47, 76, 123, \ldots$$

which have lots of interesting properties like primality testing or approximating the golden ratio geometric sequence. How do I find all the possible sequences that satisfy a recurrence, and how can I write them as a closed function of n?

Def In the examples above the recurrence relations were particularly nice! The next term was always given as a *linear combination* of previous terms. A sequence $(a_n)_{n\in\mathbb{Z}_{\geq 0}} = (a_0, a_1, a_2, \ldots)$ satisfies a *linear recurrence sequence* of degree k if a_{n+k} is a linear combination of the previous k terms. That means there are constants $c_0, \dots, c_{k-1} \in \mathbb{R}$ such that

$$a_{n+k} = c_0 a_n + c_1 a_{n+1} + \ldots + c_{k-1} a_{n+k-1} = \sum_{j \in k} c_j a_{n+j}$$

E.g. The Fibonacci sequence is a degree 2 linear recurrence relation, since the next number depends on the last two

$$f_{n+2} = 1 \cdot f_n + 1 \cdot f_{n+1}$$

E.g. The number of ternary strings length n called s_n satisfies a degree 3 linear recurrence relation. (Even though we only need 2 previous numbers, we have to go 3 back.)

$$s_{n+3} = 3s_{n+2} - s_n = -1 \cdot s_n + 0 \cdot s_{n+1} + 3 \cdot s_{n+2}$$

Question If we have a recurrence relation

$$a_{n+k} = c_0 a_n + c_1 a_{n+1} + \ldots + c_{k-1} a_{n+k-1} = \sum_{j \in k} c_j a_{n+j}$$

how could we find all the sequences that satisfy the recurrence relation? Can we express the sequences with a *closed formula* in n, rather than have to compute the entire sequence?

There's lots of methods to compute the solution set to a linear recurrence relation! Here's one method using your knowledge of linear algebra! If there's a linear combination involved a matrix is probably the right tool to handle it!

E.g. A degree 1 sequence is the easiest case.

$$a_{n+1} = 2a_n$$

Notice that inductively you can multiply by 2 and lower the index! so

$$a_n = 2a_{n-1} = 2^2 a_{n-2} = \ldots = 2^n a_0$$

But then knowing the starting term a_0 tells us the sequence!

E.g. The Fibonacci relation is degree 2. You can take and write two terms together as a vector. The the recurrence can be rewritten as a 2 by 2 matrix relation:

$$\begin{pmatrix} f_n \\ f_{n+1} \end{pmatrix} = \begin{pmatrix} f_n \\ f_n + f_{n-1} \end{pmatrix} = \begin{pmatrix} 0 & 1 \\ 1 & 1 \end{pmatrix} \begin{pmatrix} f_{n-1} \\ f_n \end{pmatrix}$$

So if lowering the index is the same as multiplying by the matrix $\begin{pmatrix} 0 & 1 \\ 1 & 1 \end{pmatrix}$ we could inductively conclude

$$\begin{pmatrix} f_n \\ f_{n+1} \end{pmatrix} = \begin{pmatrix} 0 & 1 \\ 1 & 1 \end{pmatrix}^n \begin{pmatrix} f_0 \\ f_1 \end{pmatrix}$$

So if we know the starting numbers f_0 and f_1 and if we can compute the n^{th} power of a matrix, we have determined the sequence!

Try Use the recurrence relation for s_n to conclude

$$\begin{pmatrix} s_n \\ s_{n+1} \\ s_{n+2} \end{pmatrix} = \begin{pmatrix} 0 & 1 & 0 \\ 0 & 0 & 1 \\ -1 & 0 & 3 \end{pmatrix}^n \begin{pmatrix} s_0 \\ s_1 \\ s_2 \end{pmatrix}$$

Observation One can transform an degree k linear recurrence relation into a k by k matrix equation.

$$a_{n+k} = c_0 a_n + c_1 a_{n+1} + \ldots + c_{k-1} a_{n+k-1} = \sum_{j \in k} c_j a_{n+j}$$

gives a matrix recurrence

$$\begin{pmatrix} a_n \\ a_{n+1} \\ \vdots \\ a_{n+k-1} \end{pmatrix} = \begin{pmatrix} 0 & & & \\ \vdots & I_{k-1} & & \\ 0 & & & \\ c_0 & \ldots & & c_{k-1} \end{pmatrix} \begin{pmatrix} a_{n-1} \\ a_n \\ \vdots \\ a_{n+k-2} \end{pmatrix} = A \begin{pmatrix} a_{n-1} \\ a_n \\ \vdots \\ a_{n+k-2} \end{pmatrix}$$

which has the solution

$$\begin{pmatrix} a_n \\ a_{n+1} \\ \vdots \\ a_{n+k-1} \end{pmatrix} = A^n \begin{pmatrix} a_0 \\ a_1 \\ \vdots \\ a_{k-1} \end{pmatrix}$$

for all $n \in \mathbb{Z}_{\geq 0}$, where A is the matrix with the $k-1$-sized identity matrix I_{k-1} in its upper right hand corner and with the recurrence relation coefficients written (in order!) in its last row

$$A = \begin{pmatrix} 0 & & & \\ \vdots & I_{k-1} & & \\ 0 & & & \\ c_0 & \ldots & & c_{k-1} \end{pmatrix}.$$

So a linear recurrence relation can be solved by finding a formula for the powers of a matrix! This technique can even work for linear systems with several related recurrence relations! Read Section 2 for a review[1] on the Jordan form how matrix powers can be deduced from their *spectrum*, or skip to Section 3 to see how to solve linear recurrence.

[1] maybe, you might have only seen the diagonalizable case before

2 Matrix Spectra and Powers

Computing the powers of a *diagonal* matrix is just computing the powers of the entries. Check this formula by multiplying the matrices:

$$\begin{pmatrix}1 & 0 & 0\\ 0 & 2 & 0\\ 0 & 0 & 3\end{pmatrix}^2 = \begin{pmatrix}1^2 & 0 & 0\\ 0 & 2^2 & 0\\ 0 & 0 & 3^2\end{pmatrix} = \begin{pmatrix}1 & 0 & 0\\ 0 & 4 & 0\\ 0 & 0 & 9\end{pmatrix}$$

The Spectral Theorem tells us that (after a change of coordinates) any matrix can be made diagonal, or an "almost diagonal" Jordan Cannonical Form. The *eigenvalues* are the diagonal entries, and *eigenvectors* form a nice set of coordinates. If we know how to diagonalize a matrix, computing the powers is mostly just computing the powers of the eigenvalues. For any matrix V:

$$\left(V\begin{pmatrix}2 & 0\\ 0 & 3\end{pmatrix}V^{-1}\right)^2 = V\begin{pmatrix}2 & 0\\ 0 & 3\end{pmatrix}V^{-1}V\begin{pmatrix}2 & 0\\ 0 & 3\end{pmatrix}V^{-1} = V\begin{pmatrix}2 & 0\\ 0 & 3\end{pmatrix}\begin{pmatrix}2 & 0\\ 0 & 3\end{pmatrix}V^{-1}$$

$$= V\begin{pmatrix}2^2 & 0\\ 0 & 3^2\end{pmatrix}V^{-1}$$

E.g. The Fibonacci numbers had matrix equation

$$\begin{pmatrix}f_n\\ f_{n+1}\end{pmatrix} = \begin{pmatrix}0 & 1\\ 1 & 1\end{pmatrix}^n \begin{pmatrix}f_0\\ f_1\end{pmatrix}$$

If we diagonalize $\begin{pmatrix}0 & 1\\ 1 & 1\end{pmatrix}$ into its *spectral* form or *eigendecomposition*, its characteristic polynomial is

$$\det\left(\begin{pmatrix}0 & 1\\ 1 & 1\end{pmatrix} - \lambda I\right) = \det\begin{pmatrix}-\lambda & 1\\ 1 & 1-\lambda\end{pmatrix} = \lambda^2 - \lambda - 1$$

and (use the quadratic formula) its roots are the *golden ratio*

$$\phi = \frac{1+\sqrt{5}}{2} \approx 1.618\ldots$$

and its *Galois conjugate* $-1/\phi = \frac{1-\sqrt{5}}{2}$. We can compute the eigendecomposition

$$\begin{pmatrix}0 & 1\\ 1 & 1\end{pmatrix} = \begin{pmatrix}1 & 1\\ \phi & -1/\phi\end{pmatrix}\begin{pmatrix}\phi & 0\\ 0 & -1/\phi\end{pmatrix}\begin{pmatrix}1 & 1\\ \phi & -1/\phi\end{pmatrix}^{-1}$$

so that we can compute the Fibonacci numbers [2] as

$$\begin{pmatrix}f_n\\ f_{n+1}\end{pmatrix} = \begin{pmatrix}1 & 1\\ \phi & -1/\phi\end{pmatrix}\begin{pmatrix}\phi^n & 0\\ 0 & (-1/\phi)^n\end{pmatrix}\begin{pmatrix}1 & 1\\ \phi & -1/\phi\end{pmatrix}^{-1}\begin{pmatrix}f_0\\ f_1\end{pmatrix}.$$

[2] or all possible sequences satisfying the Fibonacci relation

And now since we know $f_0 = 0$ and $f_1 = 1$ we can simplify the first row in the matrix equation to

$$f_n = \frac{\phi^n - (-1/\phi)^n}{\sqrt{5}}.$$

The Lucas numbers L_n satisfy the same recurrence, but with the different initial condition $L_0 = 2$ and $L_1 = 1$ so that they simplify to:

$$L_n = \phi^n + (-1/\phi)^n.$$

2.1 The Spectral Theorem

The Spectral Theorem says any matrix can be diagonalized if all its eigenvalues are all distinct, or semi-diagonalized into *Jordan blocks* even if the eigenvalues repeat. Some linear algebra reminders:

- Eigenvalues λ are roots of the characteristic polynomial

$$p(\lambda) = \det(A - \lambda I)$$

- A square size k by k matrix has a degree k characteristic polynomial, which has k roots (counting with repeats) in $\mathbb{C}$
- The *algebraic multiplicity* of the eigenvalue λ is the number of times λ is repeated as a root of the characteristic polynomial.
- Eigenvectors are vectors that a matrix just scales: $Av = \lambda v$ for some eigenvalue λ. You can compute eigenvectors as linearly independent solutions to

$$(A - \lambda I)v = 0$$

- The rank of $(A-\lambda I)$ is the number of linearly independent columns. $rank(A-\lambda I)$ = number of linearly independent rows = number of pivots in the reduced row eschelon form = dimension of the range
- The geometric multiplicity of eigenvalue λ is the number of linearly independent eigenvectors of A associated to λ. By the rank-nullity theorem, the geometric multiplicity of λ is $n - rank(A - \lambda I)$
- Algebraic multiplicity $\geq$ geometric multiplicity ≥ 1 for every eigenvalue. If algebraic multiplicity of $\lambda >$ geometric multiplicity of λ, then λ is a *defective* eigenvalue and A is a *defective* matrix.
- A *generalized eigenvector* is a vector w such that $(A - \lambda I)^k w = 0$ for some k. The smallest such k is the rank of the generalized eigenvector w. So eigenvectors are generalized eigenvectors of rank 1.

Spectral Theorem For every $\mathbb{R}^{n\times n}$ matrix A with j eigenvectors there is an $\mathbb{R}^n$ basis of generalized eigenvectors $V = [v_1|\dots|v_n]$ that block-diagonalize A into *Jordan Canonical form.*

$$A = V\Lambda V^{-1}$$

where Λ is a block diagonal matrix

$$\begin{pmatrix} J_{\lambda_0} & & & \mathbf{0} \\ & J_{\lambda_1} & & \\ & & \ddots & \\ \mathbf{0} & & & J_{\lambda_j} \end{pmatrix}$$

with *Jordan blocks* have a single eigenvalue λ on the diagonal and ones on the off diagonal.

$$J_\lambda = \begin{pmatrix} \lambda & 1 & 0 & \cdots & 0 \\ 0 & \lambda & 1 & \cdots & 0 \\ \vdots & \ddots & \ddots & \ddots & \vdots \\ 0 & \cdots & 0 & \lambda & 1 \\ 0 & \cdots & 0 & 0 & \lambda \end{pmatrix}$$

In particular if A has n distinct eigenvalues, then A has n eigenvectors and Λ is diagonal.

The columns of such a change of basis matrix V may be computed by forming *Jordan chains*

$$w_i \xrightarrow{A-\lambda I} w_{i-1} \xrightarrow{A-\lambda I} \ldots \xrightarrow{A-\lambda I} w_1 \xrightarrow{A-\lambda I} 0$$

of generalized vectors by iteratively choosing solutions to $(A - \lambda I)w_{k+1} = w_k$ for where $w_0 = 0$. The length i of the Jordan chain is the size of the Jordan block in Λ. The lengths of all Jordan chains for eigenvalue λ sum to the algebraic multiplicity m. The number of Jordan chains for eigenvalue λ is the geometric multiplicity of λ.

E.g. The matrix

$$A = \begin{pmatrix} 6 & -2 & -2 \\ 1 & 2 & -1 \\ 0 & 2 & 4 \end{pmatrix} = \begin{pmatrix} 2 & 2 & 1 \\ 0 & 1 & 0 \\ 2 & 0 & 0 \end{pmatrix} \begin{pmatrix} 4 & 1 & 0 \\ 0 & 4 & 1 \\ 0 & 0 & 4 \end{pmatrix} \begin{pmatrix} 2 & 2 & 1 \\ 0 & 1 & 0 \\ 2 & 0 & 0 \end{pmatrix}^{-1}$$

has eigenvalue 4 with algebraic multiplicity 3 and geometric multiplicity 1, so it has a size 3 Jordan block corresponding to the Jordan chain

$$\begin{pmatrix} 1 \\ 0 \\ 0 \end{pmatrix} \xrightarrow{A-4I} \begin{pmatrix} 2 \\ 1 \\ 0 \end{pmatrix} \xrightarrow{A-4I} \begin{pmatrix} 2 \\ 0 \\ 2 \end{pmatrix} \xrightarrow{A-4I} 0$$

Observe Jordan blocks of size k have upper diagonal powers which we can find using the Binomial Theorem and noticing $J_\lambda = \lambda I + N$ where N is the matrix with 1s one entry above the diagonal. Then N^j has just 1s in the jth entry above the diagonal and

$$J_\lambda^n = \sum_{j=0}^{n} \binom{n}{j} \lambda^{n-j} N^j = \begin{pmatrix} \lambda^n & \binom{n}{1}\lambda^{n-1} & \binom{n}{2}\lambda^{n-2} & \cdots & \cdots & \binom{n}{k-1}\lambda^{n-k+1} \\ & \lambda^n & \binom{n}{1}\lambda^{n-1} & \cdots & \cdots & \binom{n}{k-2}\lambda^{n-k+2} \\ & & \ddots & \ddots & \vdots & \vdots \\ & & & \ddots & \ddots & \vdots \\ & & & & \lambda^n & \binom{n}{1}\lambda^{n-1} \\ & & & & & \lambda^n \end{pmatrix}$$

so that even though defective matrix powers are much trickier to compute, their powers follow a definite pattern:

The Important Takeaway: Every entry of the matrix power A^n (as a function of n) is a linear combination of λ^n, $n\lambda^n$, $\ldots$, $n^{m-1}\lambda^n$ for all eigenvalues λ of multiplicity m.

Our recurrence case The linear recurrence relation

$$a_{n+k} = c_0 a_n + c_1 a_{n+1} + \ldots + c_{k-1} a_{n+k-1} = \sum_{j \in k} c_j a_{n+j}$$

gives a matrix recurrence

$$\begin{pmatrix} a_n \\ a_{n+1} \\ \vdots \\ a_{n+k-1} \end{pmatrix} = \begin{pmatrix} 0 & & \\ \vdots & I_{k-1} & \\ 0 & & \\ c_0 & \ldots & c_{k-1} \end{pmatrix} \begin{pmatrix} a_{n-1} \\ a_n \\ \vdots \\ a_{n+k-2} \end{pmatrix}$$

where the matrix has a pretty particular form!

$$A = \begin{pmatrix} 0 & & \\ \vdots & I_{k-1} & \\ 0 & & \\ c_0 & \ldots & c_{k-1} \end{pmatrix}$$

Observe that this matrix always has the recurrence coefficients written on the bottom row, and its characteristic polynomial has the same coefficients as the recurrence coefficients!

$$\det(A - \lambda I) = \lambda^k - c_{k-1}\lambda^{k-1} - \ldots - c_1\lambda - c_0 = \lambda^k - \sum_{j \in k} c_j \lambda^j.$$

Excercise In fact the eigenvectors come in a very predictable form. Check that for any eigenvalue λ if A is the form above then an eigenvector is given by

$$\begin{pmatrix} 1 \\ \lambda \\ \vdots \\ \lambda^{k-1} \end{pmatrix}$$

Putting all this together we can prove the solution set to a linear recurrence.

Theorem. If characteristic polynomial $\lambda^k - \sum_{j\in k} c_j\lambda^j$ of the degree k recurrence $a_{n+k} = \sum_{j\in k} c_j a_{n+j}$ has k distinct roots $\lambda_0, \ldots, \lambda_{k-1}$, then the solution set of the recurrence is a k-dimensional vector space spanned by the geometric sequence of each eigenvalue. I.e. for any solution sequence a_n there are constants $b_0, \ldots, b_{k-1}$ such that

$$a_n = b_0\lambda_0^n + b_1\lambda_1^n + \ldots + b_{k-1}\lambda_{k-1}^n$$

Proof. If the characteristic polynomial for a degree k recurrence has k distinct eigenvalues then

$$\begin{pmatrix} a_n \\ a_{n+1} \\ \vdots \\ a_{n+k-1} \end{pmatrix} = A^n \begin{pmatrix} a_0 \\ a_1 \\ \vdots \\ a_{k-1} \end{pmatrix}$$

$$= \begin{pmatrix} 1 & \ldots & 1 \\ \lambda_0 & \ldots & \lambda_{k-1} \\ \vdots & \ddots & \vdots \\ \lambda_0^{k-1} & \ldots & \lambda_{k-1}^{k-1} \end{pmatrix} \begin{pmatrix} \lambda_0^n & & \\ & \ddots & \\ & & \lambda_{k-1}^n \end{pmatrix} \begin{pmatrix} 1 & \ldots & 1 \\ \lambda_0 & \ldots & \lambda_{k-1} \\ \vdots & \ddots & \vdots \\ \lambda_0^{k-1} & \ldots & \lambda_{k-1}^{k-1} \end{pmatrix}^{-1} \begin{pmatrix} a_0 \\ a_1 \\ \vdots \\ a_{k-1} \end{pmatrix}$$

so then examining the first row we see

$$\begin{aligned} a_n &= \begin{pmatrix} \lambda_0^n & \lambda_1^n \ldots & \lambda_{k-1}^n \end{pmatrix} \begin{pmatrix} 1 & \ldots & 1 \\ \lambda_0 & \ldots & \lambda_{k-1} \\ \vdots & \ddots & \vdots \\ \lambda_0^{k-1} & \ldots & \lambda_{k-1}^{k-1} \end{pmatrix}^{-1} \begin{pmatrix} a_0 \\ a_1 \\ \vdots \\ a_{k-1} \end{pmatrix} \\ &= b_0\lambda_0^n + \ldots + b_{k-1}\lambda_{k-1}^n \end{aligned}$$

where the coefficients are given by the equation

$$\begin{pmatrix} 1 & \ldots & 1 \\ \lambda_0 & \ldots & \lambda_{k-1} \\ \vdots & \ddots & \vdots \\ \lambda_0^{k-1} & \ldots & \lambda_{k-1}^{k-1} \end{pmatrix} \begin{pmatrix} b_0 \\ b_1 \\ \vdots \\ b_{k-1} \end{pmatrix} = \begin{pmatrix} a_0 \\ a_1 \\ \vdots \\ a_{k-1} \end{pmatrix}$$

□

If an eigenvalue λ is repeated m times, an analysis of the Jordan block powers shows that there are solutions given by λ^n times a polynomial of degree $m-1$:

$$b'_0\lambda^n + b'_1 n\lambda^n + \ldots + b'_{m-1}n^{m-1}\lambda^n$$

3 Computing Solutions to Linear Recursion

The linear algebra in the previous section 2 tells us we can compute all possible solutions to the recurrence relation

$$a_{n+k} = \sum_{j \in k} c_j a_{n+j}$$

in the following way:

1) Plug in $a_n = x^n$ to find the characteristic polynomial is

$$0 = x^k - \sum_{j \in k} c_j x^j = \prod_{j \in k} (x - \lambda_j)$$

which has k (possibly repeated) roots $\lambda_0, \ldots, \lambda_{k-1}$ called the *eigenvalues*. Compute the eigenvalues by finding the roots of the characteristic polynomial.

2) For every eigenvalue λ, the sequence $a_n = \lambda^n$ satisfies the recurrence.
3) If the eigenvalue λ is repeated m times as a root of the characteristic polynomial, the sequences $n \mapsto \lambda^n$ and $n \mapsto n\lambda^n$ and $\ldots$ and $n \mapsto n^{m-1}\lambda^n$ satisfies the recurrence.
4) All possible solutions to the recurrence relation are given by linear combinations of these k different sequences.
5) Solve for the coefficients of the linear combination using any k known terms of the sequence.

E.g Since the Fibonacci sequence satisfies the recurrence $f_{n+2} = f_{n+1} + f_n$ we have characteristic polynomial $x^2 = x+1$ with roots $\phi_\pm = \frac{1 \pm \sqrt{5}}{2}$. So there must be constants b_0 and b_1 so that

$$f_n = b_+ \left(\frac{1+\sqrt{5}}{2} \right)^n + b_- \left(\frac{1-\sqrt{5}}{2} \right)^n$$

If we plug in $f_0 = 0$ and $f_1 = 1$ we can solve for b_+ and b_-.

$$\begin{aligned} f_0 &= 0 = b_+ + b_- \\ f_1 &= 1 = b_+ \left(\frac{1+\sqrt{5}}{2} \right) + b_- \left(\frac{1-\sqrt{5}}{2} \right) \end{aligned}$$

so that $b_\pm = \pm \frac{1}{\sqrt{5}}$.

An example with repeated roots Consider the solutions to the 5th order recurrence

$$a_{n+5} = 15a_{n+4} - 86a_{n+3} + 236a_{n+2} - 312a_{n+1} + 160a_n$$

The characteristic polynomial is (hard to factor but)

$$0 = x^5 - 15x^4 + 86x^3 - 236x^2 + 312x - 160 = (x-2)^3(x-4)(x-5)$$

The eigenvalues are 2,2,2,4,5 so the general form of the solution is

$$a_n = b_0 2^n + b_1 n 2^n + b_2 n^2 2^n + b_3 4^n + b_4 5^n$$

for any constants $b_0, \ldots, b_4 \in \mathbb{C}$. If we know the first 5 terms $a_0, \ldots, a_4$ we would solve for the b_i via

$$\begin{pmatrix} 1 & 0 & 0 & 1 & 1 \\ 2 & 2 & 2 & 4 & 5 \\ 4 & 8 & 32 & 16 & 25 \\ 8 & 24 & 216 & 64 & 125 \\ 16 & 64 & 1024 & 256 & 625 \end{pmatrix} \begin{pmatrix} b_0 \\ b_1 \\ b_2 \\ b_3 \\ b_4 \end{pmatrix} = \begin{pmatrix} a_0 \\ a_1 \\ a_2 \\ a_3 \\ a_4 \end{pmatrix}$$