Ask Yourself 2: 2 Ask 2 Yourious

1. Could I give a definition for any of these jargon words in context?

   $f = \mathcal{O}(g)$, graph, vertex, edge, walk, path, cycle, trail, circuit, Eulerian, Hamiltonian, adjacent, connected, connected-component, subgraph, complete-graph, bipartite, k-coloring, chromatic number, graph-map, graph-isomorphism, multigraph/pseudograph, multiedge, self-loop, planar, tree, forest, leaves, Prüfer code, poset, reflexive, antisymmetric, transitive, comparable, incomparable, covers, Hasse diagrams, subposet, chain, antichain, width, height, injection, surjection, permutation, derangement, the Euler totient

2. Could I state or explain the following major theorems?

   (i) The Handshaking Lemma/The Total Degree Lemma

   (ii) The Characterization of Eulerian Graphs

   (iii) Dirac's Criterion for a Hamiltonian Graph

   (iv) The Euler Characteristic of the Plane is 2

   (v) Kuratowski's Theorem

   (vi) The 4-color Theorem

   (vii) Dilworth's Theorem

   (viii) The Principle of Inclusion-Exclusion

3. Could I sort of define the *computational complexity* of an algorithm, given some basic operations of a computing model? Could I explain the distinction/relationship between **P** and **NP** and **EXP** decision problems? Could I explain the $\mathbf{P} \stackrel{?}{=} \mathbf{NP}$ question? Could I give an algorithm to answer a decision problem and estimate its $\mathcal{O}$ complexity class?

   (i) Given a list of n integers from $\{1, \ldots, 100n\}$ determine whether there are two integers x and y from the list so that x^y is a prime.

   (ii) Given a list of n integers from $\{1, \ldots, 100n\}$ determine if there is an arithmetic sequence of length k. (An arithmetic sequence $a_0, \ldots, a_{n-1}$ is a sequence with $a_{i+1} = a_i + k$ for a constant *step size* k)

4. Could I describe a polynomial procedure to solve or check solutions following decision problems? Does that show the problem to be **P** or **NP** or **EXP**?

   (i) Is an element x an element of an input set X? input x, X

   (ii) Is a set Y a subset of a set X? input Y, X

   (iii) Does a set of integers S contain a subset that sums to integer x? input S, x

   (iv) Is a graph G connected? input G

   (v) Is a graph G a tree? input G

   (vi) Is a graph H a subgraph of graph G? input G, H

(vii) Is a graph G Eulerian? input G

(viii) Is a graph G Hamiltonian? input G

(ix) Is a graph G planar? input G

(x) Is a graph G 2-colorable? input G

(xi) Is a graph G 4-colorable? input G

(xii) Is a graph G isomorphic to graph H? input G, H

(xiii) Is a graph G homeomorphic to graph H? input G, H

5. What is the greatest/least number of edges a graph on n vertices could have?

6. What is the greatest/least number of edges a tree with n vertices could have?

7. What is the greatest/least width a poset on an n element set could have?

8. What inequality could you find between the number of vertices and the number of edges of a connected planar graph if all of its cycles are at least length 5?

9. Suppose that a planar graph has 3 connected components. What inequality could you find between its number of vertices and its the number of edges?

10. Check out the graph zoo on the last page. Can you find the graphs that are disconnected? trees? Which graphs are Eulerian? Which graphs in the last row are Hamiltonian? Which graphs in the second to last row are planar?

11. Could you decide from the degree sequence of a graph if the graph is connected, a tree, Eulerian, Hamiltonian, planar, or just not at all a graph? What about for the following degree sequences?

    (i) $(4, 4, 4, 4, 2)$

    (ii) $(6, 6, 6, 6, 6, 6, 6)$

    (iii) $(1, 1, 1, 1)$

    (iv) $(8, 6, 7, 5, 3, 0, 9)$

12. Graph zoo top right: Are G and G' isomorphic? H and H'? The isomorphisms from a graph to itself are called *symmetries*. How many symmetries does K_5 have? How many symmetries does the 5-cycle graph have? How many symmetries does the 10 vertex graph on the bottom left of the graph zoo have?

13. Is every graph the Hasse diagram of a poset? G_7 in the graph zoo could be a Hasse diagram for a poset. Can you compute the width and height of G_7 and partition the vertices into the fewest possible number of chains/antichains? (Edit: This originally said G_3 which cannot be a Hasse diagram: it has triangles.)

14. Could I draw the Hasse diagram of the poset 'divides' on integers from 1 to 20 (inclusive)?

15. What must be added to the following relation so that it becomes a partial order?

$$\{(0,0),(4,4),(1,3),(1,6),(1,x),(3,5),(x,a)\}$$

What does the Hasse diagram for the poset look like?

16. What is the width of the poset of subsets of a size n set?

The Twelvefold Way:

$|\{f : k \to n\}|$

How many ways to sort k balls into n boxes?

	Arbitrary any sorting	Injective max 1 ball per box	Surjective all box gets ball
Distinct Balls Distinct Boxes	n^k	$\frac{n!}{(n-k)!}$	$n!\left\{{k \atop n}\right\}$
Identical Balls Distinct Boxes	$\binom{n+k-1}{k}$	$\binom{n}{k}$	$\binom{k-1}{n-1}$
Distinct Balls Identical Boxes	$\sum_{j=0}^{n} \left\{{k \atop j}\right\}$	1 if $k \leq n$	$\left\{{k \atop n}\right\}$
Identical Balls Identical Boxes	$p_{\leq n}(k)$	1 if $k \leq n$	$p_n(k)$

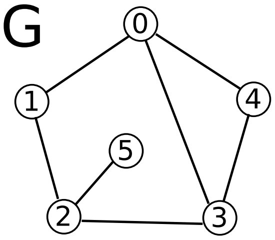
17. Could I solve counting problems by bijecting them to a ball-box-count of the 12fold way and using inclusion exclusion? E.g. 56 different delegates signed the Declaration of Independence. The delegates represented exactly 13 distinct states. How many possibilities are there for all 56 delegates to be assigned a state so that exactly 13 are represented? In how many ways can the delegates be organized into exactly disjoint 5 sets of debating factions? How many ways can we partition the delegates into 5 or fewer sets? King George III places bounties on delegates. In how many ways may he do this if he cannot exceed his budget of £5000 (but may spend less) and every bounty is a whole number of pounds? A committee of five delegates drafted the Declaration including Robert Livingston, who did not sign. Name them and give the probability of correctly guessing at random from the list of 56 delegates.

18. How many non-negative integer solutions are there to the equation $x_0 + x_2 + \ldots + x_{100} = 101$ if $x_i < 22$ for all i?

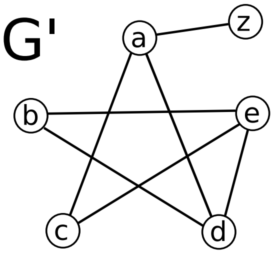
19. How many positive integers less than $1,000,000$ are not divisible by any prime less than 10?

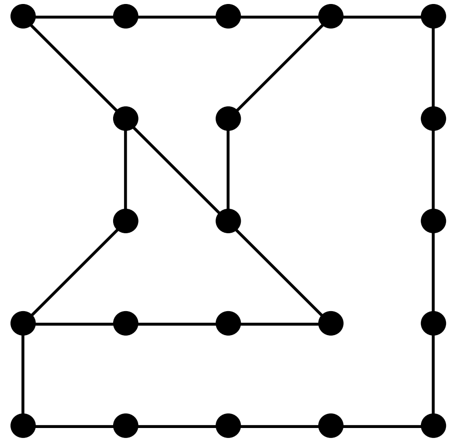
20. How many permutations of 'SO MIXED UP' move every letter? How many permutations do not contain the substrings SO, MIXED, or UP?

G

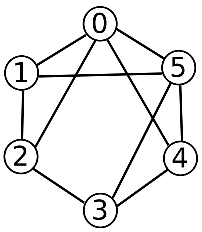


G'

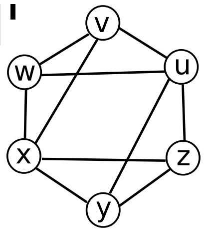




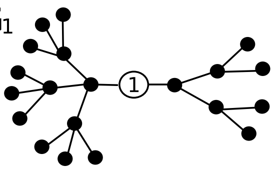
H



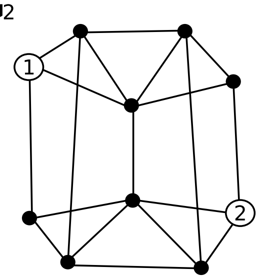
H'



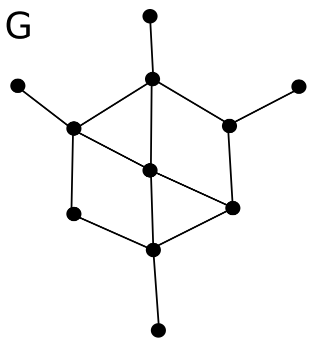
G_1



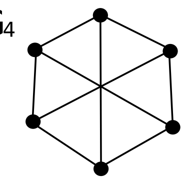
G_2



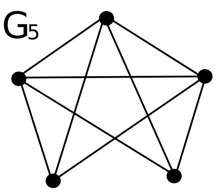
G



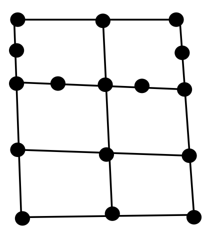
G_4



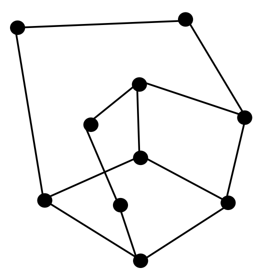
G_5



G_6



G_7



G_8