Math 1551M2 **Name:** ____________________

Fall 2017

Exam 2

20 October 2015

Time Limit: 50 Minutes

This exam contains 7 pages (including this cover page) and 5 questions. There are 40 points in total. Write explanations clearly and in complete thoughts. No calculators or notes may be used. Put your name on every page.

Grade Table

Question	Points	Score
1	9	
2	5	
3	9	
4	9	
5	8	
Total:	40	

Formal Symbols Crib Sheet

$f : A \to B$	function with domain A & codomain B	$\mathbb{N}$	natural numbers
$f \circ g$	composition of functions	$\mathbb{Z}$	integers
f^{-1}	inverse function	$\mathbb{Q}$	rational numbers
$\lim_{x\to a}$	limit as x approaches a	$\mathbb{R}$	real numbers
$\lim_{x\to a^-}$	limit from below	(a, b)	open interval a to b
$\lim_{x\to a^+}$	limit from above	$[a, b]$	closed interval a to b
$\subset$	subset of	$\in$	element of
$\cap$	intersection	$\cup$	union
$\mapsto$	maps to	f'	derivative
$\frac{d}{dx}$	derivative with respect to x		

Derivatives Crib Sheet

For constant $a \in \mathbb{R}$ and arbitrary real functions f and g

Function	Derivative	Function	Derivative
a	0	af	af'
$f+g$	$f'+g'$	fg	$f'g+fg'$
$\frac{f}{g}$	$\frac{f'g-fg'}{g^2}$	$f\circ g$	$(f'\circ g)g'$
f^{-1}	$\frac{1}{f'\circ f^{-1}}$	x^a	ax^{a-1}
a^x	$a^x \ln a$	$\log_a \lvert x\rvert$	$\frac{1}{x\ln a}$
$\sin x$	$\cos x$	$\csc x$	$-\csc x \cot x$
$\cos x$	$-\sin x$	$\sec x$	$\sec x \tan x$
$\tan x$	$\sec^2 x$	$\cot x$	$-\csc^2 x$
$\arcsin x$	$\frac{1}{\sqrt{1-x^2}}$	$\mathrm{arccsc} x$	$\frac{-1}{\lvert x\rvert\sqrt{x^2-1}}$
$\arccos x$	$\frac{-1}{\sqrt{1-x^2}}$	$\mathrm{arcsec} x$	$\frac{1}{\lvert x\rvert\sqrt{x^2-1}}$
$\arctan x$	$\frac{1}{1+x^2}$	$\mathrm{arccot} x$	$\frac{-1}{1+x^2}$
$\sinh x$	$\cosh x$	$\cosh x$	$\sinh x$

1. (a) (3 points) Find all the points where the function $f(x) = |4 - \sqrt{x}|$ is differentiable. Explain your answer.

First note that f is only defined for non-negative numbers. Since the absolute value function is not differentiable at 0, we should check differentiability when $x = 16$, since $4 - \sqrt{x} = 0$. Note that $\lim_{a\to 16^+} \frac{f(a)-f(16)}{a-16} = \frac{d}{dx}(4 - \sqrt{x})|_{x=16} = -\frac{1}{8}$ and $\lim_{a\to 16^-} \frac{f(a)-f(16)}{a-16} = \frac{d}{dx}(\sqrt{x} - 4)|_{x=16} = \frac{1}{8}$ so the graph of f has a sharp point at $x = 16$.

So f is differentiable at all the positive numbers, except for 16. In interval notation f is differentiable on $(0, 16) \cup (16, \infty)$.

(b) (6 points) Which of the following is equal to the derivative $f'(a)$ of the function f at the point $a \in \mathbb{R}$? Circle ALL that apply.

A. the slope of the line tangent to the graph of f at the point $(a, f(a))$

B. $\lim_{b\to a} \frac{f(b)-f(a)}{b-a}$

C. the angle between the x-axis and the line from $(0, 0)$ to $(a, f(a))$

D. the instantaneous rate of change of f at a

E. the average rate of change of f over the interval $(0, a)$

F. the negative reciprocal of the slope of the line normal to the graph of f at the point $(a, f(a))$

Choices A, B, D and F are true.

2. (5 points) The data table below contains some values of real functions f and g and their derivatives f' and g' evaluated at different values of x.

x	$f(x)$	$f'(x)$	$g(x)$	$g'(x)$
0	2	3	7	-5
1	3	2	5	-1
2	6	1	3	0
3	7	2	1	-1

(a) What is the derivative of the sum $f + g$ at 0?

$(f + g)'(0) = f'(0) + g'(0) = 3 - 5 = -2$

(b) What is the derivative of the product fg at 1?

$(fg)'(1) = f'(1) \cdot g(1) + f(1) \cdot g'(1) = 2 \cdot 5 + 3 \cdot -1 = 7$

(c) What is the derivative of the composition $f \circ g$ at 3?

$(f \circ g)'(3) = f'(g(3)) \cdot g'(3) = f'(1) \cdot -1 = -2$

(d) Remember that the inverse function of g is written g^{-1}. Find a point where g^{-1} is not differentiable.
Since $g'(2) = 0$ it must be that $(g^{-1})'(3) = \frac{1}{g'(g^{-1}(2))} = \frac{1}{g'(2)}$ does not exist. So g^{-1} is not differentiable at 3.

(e) What is the average rate of change in g over the interval $(0, 3)$?

$$\frac{g(3) - g(0)}{3 - 0} = \frac{1 - 7}{3 - 0} = -2$$

3. (a) (3 points) Compute the derivative

$$\frac{d}{dx}\left(\sin(x)\sin(x)\sin(x)\right) =$$

$$\frac{d}{dx}(\sin x)^3 = 3(\sin x)^2 \cos x$$

(b) (3 points) Compute the derivative

$$\frac{d}{dx}\left(\sin\left(\sin\left(\sin(x)\right)\right)\right) =$$

$$\frac{d}{dx}\left(\sin\left(\sin\left(\sin(x)\right)\right)\right) = \cos\left(\sin\left(\sin(x)\right)\right)\cos\left(\sin x\right)\cos(x)$$

(c) (3 points) Compute the derivative

$$\frac{d}{dx}\left(\frac{\arctan(x)}{\ln(x)}\right) =$$

$$\frac{d}{dx}\left(\frac{\arctan(x)}{\ln(x)}\right) = \frac{\frac{1}{1+x^2}\ln x - \arctan(x)\cdot\frac{1}{x}}{(\ln x)^2}$$

4. Suppose that a object moves vertically with height $h(t) = (3t - \frac{1}{3t})^2$ in meters, where t is measured in time.

   (a) (3 points) Find the instantaneous velocity of the object at any time t.

   The instantaneous velocity is

$$h'(t) = 2\left(3t - \frac{1}{3t}\right)\left(3 + \frac{1}{3t^2}\right) = 18t - \frac{2}{9t^3}$$

   (b) (3 points) Find the average velocity of the object between time 1/3 sec and 1 sec.

$$\frac{h(1) - h(1/3)}{1 - 1/3} = \frac{(3 - 1/3)^2 - 0}{2/3} = \frac{32}{3}$$

   (c) (3 points) Find the acceleration of the object at any time t.

   The acceleration is the second derivative of position

$$h''(t) = \frac{d}{dt}(18t - \frac{2}{9t^3}) = 18 - \frac{2}{3t^4}$$

5. (8 points) Consider the curve in the x-y plane defined by all the points (x, y) that satisfy the equation

$$\frac{y+2x}{3} = e^{-(y-x)^2}$$

Compute the equation of the tangent and normal lines to the curve at the point $(1, 1)$.

If we differentiate with respect to x on of both sides of the equation we get

$$\frac{y'+2}{3} = e^{-(y-x)^2} \cdot -2(y-x) \cdot (y'-x)$$

which we want to solve for y' when $x = y = 1$. Plugging in $x = y = 1$ we have

$$\frac{y'+2}{3} = e^{-0^2} \cdot 0 \cdot (y'-1) = 0$$

so $y' = 2$.

Then the tangent line has equation

$$y - 1 = -2(x-1)$$

The normal line is perpindicular to the tangent line, so it's slope is $-\frac{1}{-2} = \frac{1}{2}$ and it passes through the same point $(1, 1)$ so it has equation

$$y - 1 = \frac{1}{2}(x-1)$$