Final Exam Review Exercise Set A, Math 1551, Fall 2017

This review set gives a **list of topics** that we explored throughout this course, as well as a few practice problems at the end of the document. A complete list of **what you are expected to be able to do** (learning objectives) is in the learning objectives document, on T^2. Don't forget:

- **Topics** indicate what material is explored in a course. They do not articulate expectations.
- **Learning objectives** are statements that articulate what students are expected to be able to do in a course.

For example, **indefinite integrals** is a topic. A **learning objective** associated with that topic would be: apply indefinite integrals to solve differential equations and initial value problems.

Suggestions for Preparing for the Final

Students are encouraged to prepare by:

1. completing all of the practice review sets,
2. completing additional problems from the textbook,
3. spend most of their preparing by solving problems,
4. solving lots of problems,
5. asking questions during office hours and/or piazza,
6. spending some time studying with other students,
7. start studying as early possible.

I also recommend focusing your time studying those topics you struggle with most, and spending less time reading the textbook and making study notes at this point. Spend most of your time solving problems.

The MML Study Plan

If you would like to study by solving problems that have solutions, the MML Study Plan has hundreds of problems you can solve. MML will tell you if your work is correct and offers a few different study aids. To access problems for a specific textbook section:

1. navigate to mymathlab.com and log in
2. select your Math 1551 course
3. select Thomas (the online textbook)
4. select a chapter
5. select a section
6. click study plan

Chapter 1 Functions

The material in this chapter is background that you may have encountered in pre-requisite courses. Most (possibly all) questions on your final exam will require familiarity with Chapter 1 material.

Functions and Their Graphs

Topics: functions and the vertical line test,domain and range, even/odd functions, increasing/decreasing intervals, piecewise functions. This material is especially useful for curve sketching.

Combining Functions; Shifting and Scaling Graphs

Topics: sketch functions (using shifting, scaling, reflections), and compose functions (find the domain and range of the composition).

Trigonometric Functions

Topics: degrees and radians, trigonometric functions and identities, graphing and evaluating $\sin x, \cos x$, $\tan x$, $\cot x, \sec x$, and $\csc x$. What are the values of each of those functions at $x = 0, \pi/6, \pi/4, \pi/3, \pi/2$? (If you know the values of $\sin x$ and $\cos x$ at those points, you can get all the others.) Can you use transformations to sketch the graphs of functions like $\sin(2x), 3\cos x, \cos(x - 3\pi/2), |\sin(\pi x)|$, etc.?

Inverse Functions and Logarithms

Topics: the inverse of a function, the relationship between the graph of the inverse function and the graph of the original function, inverse trig functions, exponential function and logarithm functions. You should be able to sketch e^x and $\ln x$, and use the properties of exponential/logarithm functions.

Chapter 2 Limits and Continuity

A few strategies for computing limits: if you encounter indeterminate forms such as $0/0$, ∞/∞, $0 \cdot \infty$ or $\infty - \infty$, try the following strategies.

1. If you are dealing with a rational function such as $\lim_{x \to 3} \frac{x^2 - 3x}{x^2 - 9}$, try to factorize the numerator and denominator.
2. If it include some kind of square root, like $\lim_{x \to 4} \frac{x - 4}{\sqrt{x} - 2}$ or $\lim_{x \to \infty} x - \sqrt{x^2 - 1}$, often it is helpful to multiply by its conjugate.
3. An important limit is $\lim_{x \to 0} \frac{\sin x}{x} = 1$. For some limits involving the trig functions, you might be able to manipulate the expression and make use of this limit.
4. For the ∞/∞ type indeterminate form, usually it is helpful to factor out the leading term in both the numerator and denominator. For example: $\lim_{x \to \infty} \frac{4x^2 - 1}{\sqrt{3x^4 + 5}}$.

The sandwich theorem is particularly useful for examples such as $\lim_{x \to 0} x \sin(\frac{1}{x})$ and $\lim_{x \to \infty} \frac{\sin x}{e^x}$ where the limit laws and the above tricks do not apply.

2.1 Rates of Change and Tangents to Curves

Topics: estimating the rate of change of a function, the equation of a tangent line to a function at a point.

2.2 Limit of a Function and Limit Laws

Topics: Limit laws, limits of Functions, the Sandwich (or Squeeze) Theorem

2.3 The Precise Definition of a Limit

Topics: The precise definition of limit.

2.4 One-Sided Limits

Topics: One-sided limits of Functions, limits involving $\frac{\sin \theta}{\theta}$ and $\frac{\cos \theta - 1}{\theta}$

2.5 Continuity

Topics: continuity, the intermediate value theorem (IVT)

2.6 Limits Involving Infinity; Asymptotes of Graphs

Topics: limits at infinity, horizontal, oblique, and vertical asymptotes.

Chapter 3 Differentiation

Section 3.1: Tangents and the Derivative at a Point

Topics: Review average and instantaneous rate of change, the derivative, computation and interpretation of the derivative.

Section 3.2: The Derivative as a Function

Topics: the derivative of a function, sketching the derivative of a function, differentiability.

Important: if f is differentiable at $x = c$, then f must be continuous at $x = c$. But f could be continuous at $x = c$ without being differentiable at $x = c$.

Section 3.3: Differentiation Rules

Topics: the derivative rules (addition, subtraction, product, quotient, chain rule), solve equations involving derivatives (for example, to locate points on a graph where the tangent line has a particular slope), higher derivatives, derivatives of trig functions, exponential functions and logarithm functions.

Sections 3.4 The Derivative as a Rate of Change

Topics: Velocity, speed, acceleration.

Section 3.5, 3.6, 3.7: The Derivative as a Rate of Change, Derivatives of Trigonometric Functions, The Chain Rule

Topics: Derivatives of the trigonometric functions, the chain rule, implicit differentiation, normal lines.

Sections 3.8: Derivatives of Inverse Functions and Logarithms

Topics: exponential functions and logarithm functions.

Section 3.10: Related Rates

Topics: related rates. Solving rate problems tend to involve the following sequence of steps.

1. Read the question.
2. Draw a diagram.
3. Introduce variables.
4. Construct an equation.
5. Calculate derivative at a point.
6. Express answer to question using appropriate units.

Please express final answer with units.

Section 3.11: Linearization and Differentials

Topics: Linear approximation and differentials. You should know how to construct linear approximations and differentials of a function.

Chapter 4 Applications of Derivatives

Extreme Values of Functions, Monotonic Functions, the First Derivative Test

Topics: Critical points, absolute extrema and local extrema, identifying where functions are increasing and where they are decreasing, the first derivative test.

The Mean Value Theorem

Topics: Rolle's Theorem, The Mean Value Theorem (MVT), consequences of the MVT (theorems).

Concavity and Curve Sketching

Section 4.4 covers the 2nd derivative test, concavity, and curve sketching.

- Second derivative test: for a differentiable function f, how do we determine whether a critical point $x = c$ is local max or min? There are two ways: you can either check how f' changes sign from the left to the right of $x = c$, or you can also look at the sign of $f''(c)$. However the second method won't give you any information if you happen to have $f''(c) = 0$.
- Concavity. To determine whether a function is concave up/down on an interval, one could check the sign of $f''(x)$. If a function changes its concavity at $x = c$, then it is called an inflection point. The concavity must change in order for a point to be an inflection point. An inflection point is NOT just where $f''(x) = 0$, the concavity must change.
- Curve sketching. Given a function f, you will be asked to identify all the local max/min, intervals in which it is increasing/decreasing, intervals in which it is concave up/down, symmetry (even/odd/neither), intercepts, domain and range, and asymptotes (if any). You will need to use some/all of this information to sketch a graph of a function.

Applied Optimization

These are word problems that will require you to use what we learned before to identify a max or min of a function that you have constructed.

Newton's Method

Topic: Newton's Method for solving $f(x) = 0$, given an initial starting point, x_0.

Antiderivatives

Topics: antiderivatives, indefinite integrals, initial value problems. You need to have memorized the antiderivatives of the functions listed in the lecture slides (which includes x^n, e^x, $\sin x$, $\cos x$, $\frac{1}{x}$, plus a few others, and their combinations).

Practice Problems

Ex 1.1 Let $f(x) = \ln(x^2+1)$. Determine the domain and range of $f(x)$, and if the function is even or odd.

Ex 1.2. True or false: If $g(x)$ is an odd and continuous function defined for all values of x, then we must have $g(0) = 0$.

Ex 2.1. True or false: If both $\lim_{x\to a} f(x)$ and $\lim_{x\to a} g(x)$ exist, then $\lim_{x\to a} \dfrac{f(x)}{g(x)}$ must exist.

Ex 2.2. Evaluate the following limits: (a) $\lim_{x\to 0} \dfrac{x(x+3)}{\tan 2x}$; (b) $\lim_{x\to 3} \dfrac{x^2-9}{\sqrt{x+1}-2}$.

Ex 2.3. True or false: $\lim_{x\to\infty} \dfrac{\sin x}{x} = 1$.

Ex 2.4. Find $\lim_{x\to\infty} \dfrac{4x^2-1}{\sqrt{3x^4+5}}$.

Ex 2.5. Give an example of a function f defined on $(-\infty, \infty)$, such that $\lim_{x\to 0^+} f(x)$ exists but $\lim_{x\to 0} f(x)$ doesn't exist.

Ex 2.6. If $f(x) = \begin{cases} x^2+a & x > 1 \\ 2x & x \le 1 \end{cases}$, then how should we choose a in order for f to be a continuous function?

Ex 2.7. Find all vertical and horizontal asymptotes of $f(x) = \dfrac{\sqrt{x}-1}{x^2-x}$.

Ex 3.1. Use the definition to compute the derivative of $f(x) = \sqrt{x+1}$.

Ex 3.2. For $f(x) = |x^{1/3}|$, is it continuous at $x = 0$? Is it differentiable at $x = 0$?

Ex 3.4. Find the derivative of $f(x) = \ln(\tan^{-1}(x^2))$.

Ex 3.5. What is the normal line to the graph of $x^2 + xy^4 = 2$ at the point $(1, 1)$?

Ex 3.6. A train, starting at 11am, travels east at 45mph while another, starting at noon from the same point, travels south at 30mph. How fast are they separating at 3pm?

Ex 3.7. Approximate the number $\sqrt[4]{1.1}$.

Ex 4.1. True or false: Every continuous function must have an absolute maximum in $(-\infty, \infty)$.

Ex 4.2. If $f(x) = \dfrac{x-1}{x^2+1}$, find all intervals such that f is increasing/decreasing.

Ex 4.3. Show that $x^5 + e^x = 4$ has exactly one solution.

Ex 4.4. True or false: If $f'(x) = 0$ for all x, then $f(x)$ must be a constant.

Ex 4.5. True or false: The function $f(x) = \ln(\cos x)$ has a local max at $x = 0$.

Ex 4.6. Sketch the graph of $f(x) = x^3 + \dfrac{3}{x}$.

Ex 4.7. Find the dimensions of the rectangle of largest area that can be inscribed in a semicircle of radius 1.

Ex 4.8. You are making a square-bottomed box with no top and want to maximize the total volume that it can hold while using no more than 600 square inches of material. What's the biggest box you can make?

Ex 4.9. Let $f(x) = x^3 + x - 1$. Use Newton's Method to approximate the value of the x-intercept. Start with $x_0 = 0$ and perform two iterations. (i.e. Find x_2).

Ex 4.10. Calculate the antiderivative of $f(x) = \cos x + 3x^3 - \frac{2}{1+x^2}$.

Solutions

Ex 1.1 Let $f(x) = \ln(x^2+1)$. Determine the domain and range of $f(x)$, and if the function is even or odd.
Solution. This function is even since $f(-x) = \ln((-x)^2+2) = f(x)$. Its domain is $(-\infty, \infty)$, and its range is $[\ln 2, +\infty)$.
Ex 1.2. True or false: If $g(x)$ is an odd and continuous function defined for all values of x, then we must have $g(0) = 0$.
Solution. This is true. To convince yourself that it is, try drawing an odd function that does not pass through the origin. But $g(x)$ being odd means $g(-x) = -g(x)$ for all x, and plugging in $x = 0$ gives us $g(0) = -g(0)$, hence $2g(0) = 0$, which implies $g(0) = 0$.

Ex 2.1. True or false: If both $\lim_{x\to a} f(x)$ and $\lim_{x\to a} g(x)$ exist, then $\lim_{x\to a} \frac{f(x)}{g(x)}$ must exist.
Solution. This is false. For example, if $f(x) = x$ and $g(x) = x^2$, then we have both $\lim_{x\to 0} f(x)$ and $\lim_{x\to 0} g(x)$ exist, but $\lim_{x\to 0} \frac{f(x)}{g(x)} = \lim_{x\to 0} \frac{x}{x^2} = \lim_{x\to 0} \frac{1}{x}$ doesn't exist.

Ex 2.2. Evaluate the following limits: (a) $\lim_{x\to 0} \frac{x(x+3)}{\tan 2x}$; (b) $\lim_{x\to 3} \frac{x^2-9}{\sqrt{x+1}-2}$.
Solution.

$$\lim_{x\to 0} \frac{x(x+3)}{\tan 2x} = \lim_{x\to 0} \frac{x(x+3)}{\sin 2x/\cos 2x} = \lim_{x\to 0} \frac{x(x+3)\cos 2x}{\sin 2x} = \lim_{x\to 0} \frac{x}{\sin 2x} \cdot (x+3)\cos 2x = \frac{1}{2}\cdot 3 = \frac{3}{2}.$$

$$\lim_{x\to 3} \frac{x^2-9}{\sqrt{x+1}-2} = \lim_{x\to 3} \frac{(x+3)(x-3)(\sqrt{x+1}+2)}{(\sqrt{x+1}-2)(\sqrt{x+1}+2)} = \lim_{x\to 3} \frac{(x+3)(x-3)(\sqrt{x+1}+2)}{(x-3)} = 24.$$

Ex 2.3. True or false: $\lim_{x\to\infty} \frac{\sin x}{x} = 1$.

Solution. This is false. Do not mistaken it with $\lim_{x\to 0} \frac{\sin x}{x} = 1$! (In these two expressions, the limits that x is approaching are different). Actually, we have $\lim_{x\to\infty} \frac{\sin x}{x} = 0$. This is because $|\sin x| \le 1$, hence for all positive x we have

$$-\frac{1}{x} \le \frac{\sin x}{x} \le \frac{1}{x}.$$

And since we know both $-\frac{1}{x}$ and $\frac{1}{x}$ approach 0 as $x \to \infty$, by sandwich theorem, we have $\lim_{x\to\infty} \frac{\sin x}{x} = 0$.

Ex 2.4. Find $\lim_{x\to\infty} \frac{4x^2-1}{\sqrt{3x^4+5}}$.
Solution.

$$\lim_{x\to\infty} \frac{4x^2-1}{\sqrt{3x^4+5}} = \lim_{x\to\infty} \frac{x^2(4-\frac{1}{x^2})}{x^2\sqrt{3+\frac{5}{x^4}}} = \frac{4}{\sqrt{3}}.$$

Ex 2.5. Give an example of a function f defined on $(-\infty, \infty)$, such that $\lim_{x\to 0^+} f(x)$ exists but $\lim_{x\to 0} f(x)$ doesn't exist.
Solution. For example, $f(x) = \begin{cases} 0 & x \ge 0 \\ 1 & x < 0 \end{cases}$ is such an example: there we have $\lim_{x\to 0^+} f(x) = 0$ but $\lim_{x\to 0} f(x)$ doesn't exist.

Ex 2.6. If $f(x) = \begin{cases} x^2 + a & x > 1 \\ 2x & x \le 1 \end{cases}$, then how should we choose a in order for f to be a continuous function?
Solution. In order for f to be continuous at $x = 1$, we need to make sure that $\lim_{x \to 1} f(x) = f(1)$. In order for the left-hand limit of f match the right-hand limit of f, we have $1^2 + a = 2 \cdot 1$, hence $a = 1$.

Ex 2.7. Find all vertical and horizontal asymptotes of $f(x) = \dfrac{\sqrt{x} - 1}{x^2 - x}$.
Solution. Note that the denominator is $x(x-1)$, so $f(x)$ is undefined when $x = 0$ and $x = 1$. We have $\lim_{x \to 0^+} f(x) = \infty$ and $\lim_{x \to 0^-} f(x) = -\infty$, so $x = 0$ is a vertical asymptote. However, $x = 1$ is NOT a vertical asymptote since we have $\lim_{x \to 1} f(x) = \lim_{x \to 1} \dfrac{(\sqrt{x}-1)(\sqrt{x}+1)}{x(x-1)(\sqrt{x}+1)} = \dfrac{1}{2}$, which is a finite limit.
As for horizontal asymptote, we have

$$\lim_{x \to \infty} \frac{\sqrt{x} - 1}{x^2 - x} = \lim_{x \to \infty} \frac{\sqrt{x}\sqrt{1 - \frac{1}{x}}}{x^2(1 - \frac{1}{x})} = \lim_{x \to \infty} \frac{\sqrt{1 - \frac{1}{x}}}{x^{3/2}(1 - \frac{1}{x})} = \frac{1}{\infty} = 0.$$

So $y = 0$ is the only horizontal asymptote. (There is no need to check the limit $x \to \infty$, since the domain of this function is $x \ge 0$.)

Ex 3.1. Use the definition to compute the derivative of $f(x) = \sqrt{x+1}$.
Solution. By definition, we have

$$\begin{aligned} f'(x) &= \lim_{h \to 0} \frac{f(x+h) - f(x)}{h} \\ &= \lim_{h \to 0} \frac{\sqrt{x+h+1} - \sqrt{x+1}}{h} \\ &= \lim_{h \to 0} \frac{(\sqrt{x+h+1} - \sqrt{x+1})(\sqrt{x+h+1} + \sqrt{x+1})}{h(\sqrt{x+h+1} + \sqrt{x+1})} \\ &= \lim_{h \to 0} \frac{h}{h(\sqrt{x+h+1} + \sqrt{x+1})} \\ &= \frac{1}{2\sqrt{x+1}}. \end{aligned}$$

Ex 3.2. For $f(x) = |x^{1/3}|$, is it continuous at $x = 0$? Is it differentiable at $x = 0$?
Solution. It is continuous at $x = 0$ since $\lim_{x \to 0} f(x) = 0 = f(0)$. However, f is not differentiable at $x = 0$. To see this, note that $f'(x) = \frac{1}{3}x^{-2/3}$ for $x > 0$, and $f'(x) = -\frac{1}{3}x^{-2/3}$ for $x < 0$. These are approaching $+\infty$ and $-\infty$ respectively as $x \to 0$, so f is not differentiable at $x = 0$.

$f^{(n)}(x) = (-1)^n (x - n) e^{-x}$.

Ex 3.4. Find the derivative of $f(x) = \ln(\tan^{-1}(x^2))$.
Solution. Using the chain rule, we have

$$f'(x) = \frac{1}{\tan^{-1}(x^2)} \cdot \frac{1}{1 + (x^2)^2} \cdot 2x = \frac{2x}{\tan^{-1}(x^2)(1 + x^4)}.$$

Ex 3.5. What is the normal line to the graph of $x^2 + xy^4 = 2$ at the point $(1, 1)$?

Solution. Let us take $\frac{d}{dx}$ on both sides of the equation (and use product rule and chain rule here):

$$2x + y^4 + x \cdot 4y^3 \frac{dy}{dx} = 0,$$

hence

$$\frac{dy}{dx} = -\frac{2x + y^4}{4xy^3},$$

which is equal to $-\frac{3}{4}$ at $(1, 1)$. As a result, the slope of the *normal* line through (1,1) is $\frac{-1}{(-3/4)} = \frac{4}{3}$, and the equation of normal line is

$$y = \frac{4}{3}(x - 1) + 1 = \frac{4}{3}x - \frac{1}{3}.$$

Ex 3.6. A train, starting at 11am, travels east at 45mph while another, starting at noon from the same point, travels south at 30mph. How fast are they separating at 3pm?

Solution. At 3pm, we know that the first train is 180mi to the east of the starting point, and the second train is 90mi to the south of the starting point. Let $z(t)$ denote their distance. We have $z^2 = x^2 + y^2$ (where $dx/dt = 45$ and $dy/dt = 30$), hence at 3pm, we have

$$2z\frac{dz}{dt} = 2x\frac{dx}{dt} + 2y\frac{dy}{dt},$$

which gives

$$2\sqrt{90^2 + 180^2}\frac{dz}{dt} = 2 \cdot 180 \cdot 45 + 2 \cdot 90 \cdot 30,$$

thus they are separating with velocity $24\sqrt{5}$ mph.

Ex 3.7. Approximate the number $\sqrt[4]{1.1}$.

Solution. Let $f(x) = \sqrt[4]{x}$, so we have $f(1) = 1$ and our goal is to approximate $f(1.1)$. Note that $f'(1) = \frac{1}{4}1^{-3/4} = \frac{1}{4}$, so the linear approximation of $f(x)$ around $x = 1$ is $L(x) = \frac{1}{4}(x - 1) + 1$. Making use of this, we have $f(1.1) \approx \frac{1}{4}(1.1 - 1) + 1 = 1.025$.

Ex 4.1. True or false: Every continuous function must have an absolute maximum in $(-\infty, \infty)$.

Solution. This is false. For example the function $f(x) = x$ does not have any absolute maximum or minimum in $(-\infty, \infty)$.

Ex 4.2. If $f(x) = \frac{x - 1}{x^2 + 1}$, find all intervals such that f is increasing/decreasing.

Solution. Taking the derivative of f and simplifying the expression, we have $f'(x) = \frac{-x^2 + 2x + 1}{(x^2 + 1)^2}$. Therefore it has critical points $x = 1 \pm \sqrt{2}$, and we have $f'(x) > 0$ in $(1 - \sqrt{2}, 1 + \sqrt{2})$, and $f'(x) < 0$ in $(-\infty, 1-\sqrt{2})$ and $(1+\sqrt{2}, \infty)$. As a result, f is increasing in $(1-\sqrt{2}, 1+\sqrt{2})$ and decreasing in $(-\infty, 1-\sqrt{2})$ and $(1 + \sqrt{2}, \infty)$.

Ex 4.3. Show that $x^5 + e^x = 4$ has exactly one solution.

Solution. Note that $f(x)$ is a continuous function, and $f(0) = 0 + e^0 = 1 < 4$, $f(2) = 2^5 + e^2 > 4$. By intermediate value theorem, $f(x) = 4$ must have at least one solution in $(0, 2)$. However, we also have that $f'(x) = 5x^4 + e^x > 0$ for all x, which means $f(x)$ does not have any critical points in $(-\infty, \infty)$. So $f(x) = 4$ must have no more than one solution in $(-\infty, \infty)$, since otherwise $f(x)$ would have a critical point by Rolle's Theorem. Combining these together gives us $f(x) = 4$ has exactly one solution.

Ex 4.4. True or false: If $f'(x) = 0$ for all x, then $f(x)$ must be a constant.
Solution. This is true. Take any interval (a, b). By mean value theorem, we have $f(b) - f(a) = (b-a) \cdot f'(c)$ for some c in (a, b), but since $f'(x) = 0$ for all x, we have $f(b) - f(a) = 0$ for all a, b, which means $f(x)$ is a constant.

Ex 4.5. True or false: The function $f(x) = \ln(\cos x)$ has a local max at $x = 0$.
Solution. This is true. We have $f'(x) = \frac{1}{\cos x} \cdot (-\sin x) = -\tan x$, which is positive in $(-\pi/2, 0)$ and negative in $(0, \pi/2)$. Hence f is increasing in $(-\pi/2, 0)$ and decreasing in $(0, \pi/2)$, which means that $x = 0$ is a local max.

Ex 4.6. Sketch the graph of $f(x) = x^3 + \dfrac{3}{x}$.
Solution. I will not post the graph here, but the steps are as follows: First, the domain is $x \neq 0$. In its domain, we have $f'(x) = \dfrac{3(x^4-1)}{x^2} = \dfrac{3(x-1)(x+1)(x^2+1)}{x^2}$, hence f is increasing in $(1, \infty)$ and $(-\infty, 1)$, and decreasing in $(-1, 0)$ and $(0, 1)$. As a result, f has local min at the point $(1, 4)$, and local max at the point $(-1, -4)$, and you should mark these points on your graph.
As for concavity, we have $f'' = 6x + \dfrac{6}{x^3} = \dfrac{6(x^4+1)}{x^3}$, hence f is concave up in $(0, \infty)$ and concave down in $(-\infty, 0)$. (However $x = 0$ is NOT an inflection point since f is undefined at $x = 0$).
As for asymptotes and limiting behavior, f has vertical asymptote at $x = 0$, and more precisely we have $\lim_{x \to 0^+} f(x) = +\infty$ and $\lim_{x \to 0^-} f(x) = -\infty$. We also have $\lim_{x \to \infty} f(x) = +\infty$ and $\lim_{x \to -\infty} f(x) = -\infty$.

Ex 4.7. Find the dimensions of the rectangle of largest area that can be inscribed in a semicircle of radius 1.
Solution. The equation for a unit circle is $x^2 + y^2 = 1$, hence for the upper semi-circle we have $y = \sqrt{1-x^2}$. If the rectangle is inscribed in the semi-circle and its top right corner is at (x, y), then it would have length $2x$ and width y, hence its area would be $2xy$. So our goal is to maximize $2xy$ subject to the constraint $y = \sqrt{1-x^2}$. Plugging $y = \sqrt{1-x^2}$ into the area function, we are looking for the maximum of

$$f(x) = 2x\sqrt{1-x^2}$$

with domain $0 \leq x \leq 1$.
We have

$$f'(x) = 2\sqrt{1-x^2} + 2x \cdot \frac{1}{2\sqrt{1-x^2}} \cdot 2x = \frac{2-4x^2}{\sqrt{1-x^2}},$$

hence the only critical point in $[0, 1]$ is $x = \frac{1}{\sqrt{2}}$, at which we have $f(1/\sqrt{2}) = 2 \cdot \frac{1}{\sqrt{2}}\sqrt{1 - 1/2} = 1$. As for boundary values, we have $f(0) = 0$ and $f(1) = 0$, hence the function has its absolute maximum value 1 when $x = 1/\sqrt{2}$. In other words, the dimensions of the rectangle with the largest area is: length = $\sqrt{2}$, width = $1/\sqrt{2}$.

Ex 4.8. You are making a square-bottomed box with no top and want to maximize the total volume that it can hold while using no more than 600 square inches of material. What's the biggest box you can make?
Solution. Let x be the side length of the bottom of the box, and h be the height of the box. Our goal is to maximize the volume x^2h subject to the constraint $x^2 + 4xh \leq 600$. This gives $h \leq \dfrac{600-x^2}{4x}$, and clearly we should set h to be exactly equal this to make the volume as large as possible. Plugging in $h = \dfrac{600-x^2}{4x}$ into the volume formula, we know that we are looking for the maximum of the function

$$f(x) = x^2 \cdot \frac{600-x^2}{4x} = \frac{x(600-x^2)}{4}$$

in the domain $0 < x \leq \sqrt{600}$.

We have $f'(x) = 150 - \dfrac{3x^2}{4}$, hence the only critical point is $x = 10\sqrt{2}$. At this point we have $f(10\sqrt{2}) = 1000\sqrt{2}$. As for boundary points, we have $f(\sqrt{600}) = 0$, and $\lim\limits_{x \to 0^+} f(x) = 0$. So the maximum volume is $1000\sqrt{2}$, and it is achieved when the side length of the bottom is $10\sqrt{2}$, and the height is $5\sqrt{2}$.

Ex 4.9. Let $f(x) = x^3 + x - 1$. Use Newton's Method to approximate the value of the x-intercept. Start with $x_0 = 0$ and perform two iterations. (i.e. Find x_2).

Solution. Note that $f'(x) = 3x^2 + 1$. At $x_0 = 0$, we have $f(x_0) = f(0) = -1$ and $f'(x_0) = 1$. Hence $x_1 = x_0 - \dfrac{f(x_0)}{f'(x_0)} = 1$. At $x_1 = 1$, we have $f(x_1) = f(1) = 1$ and $f'(1) = 4$, hence $x_2 = x_1 - \dfrac{f(x_1)}{f'(x_1)} = 1 - \dfrac{1}{4} = \dfrac{3}{4}$.

Ex 4.10. Calculate the antiderivative of $f(x) = \cos x + 3x^3 - \frac{2}{1+x^2}$.

Solution. The antiderivative is $F(x) = \sin x + \dfrac{3}{4}x^4 - 2\tan^{-1}(x) + C$.