Math 1551-G **Name:** ____________________________

Fall 2015

Exam 4

20 November 2015

Time Limit: 50 Minutes

This exam contains 7 pages (including this cover page) and 5 questions. There are 48 points in total. Write explanations clearly and in complete thoughts. No calculators or notes may be used. Put your name on every page.

Grade Table

Question	Points	Score
1	9	
2	9	
3	12	
4	9	
5	9	
Total:	48	

Formal Symbols Crib Sheet

$f : A \to B$	function with domain A & codomain B	$\mathbb{N}$	natural numbers
$f \circ g$	composition of functions	$\mathbb{Z}$	integers
f^{-1}	inverse function	$\mathbb{Q}$	rational numbers
$\lim_{x\to a}$	limit as x approaches a	$\mathbb{R}$	real numbers
$\lim_{x\to a^-}$	limit from below	(a, b)	open interval a to b
$\lim_{x\to a^+}$	limit from above	$[a, b]$	closed interval a to b
$\subset$	subset of	$\in$	element of
$\cap$	intersection	$\cup$	union
$\mapsto$	maps to	f'	derivative
$\frac{d}{dx}$	derivative with respect to x		

Derivatives Crib Sheet

For constant $a \in \mathbb{R}$ and arbitrary real functions f and g

Function	Derivative	Function	Derivative
a	0	af	af'
$f+g$	$f'+g'$	fg	$f'g+fg'$
$\frac{f}{g}$	$\frac{f'g-fg'}{g^2}$	$f \circ g$	$(f' \circ g)g'$
f^{-1}	$\frac{1}{f' \circ f^{-1}}$	x^a	ax^{a-1}
a^x	$a^x \ln a$	$\log_a \lvert x\rvert$	$\frac{1}{x \ln a}$
$\sin x$	$\cos x$	$\csc x$	$-\csc x \cot x$
$\cos x$	$-\sin x$	$\sec x$	$\sec x \tan x$
$\tan x$	$\sec^2 x$	$\cot x$	$-\csc^2 x$
$\arcsin x$	$\frac{1}{\sqrt{1-x^2}}$	$\operatorname{arccsc} x$	$\frac{-1}{\lvert x\rvert\sqrt{x^2-1}}$
$\arccos x$	$\frac{-1}{\sqrt{1-x^2}}$	$\operatorname{arcsec} x$	$\frac{1}{\lvert x\rvert\sqrt{x^2-1}}$
$\arctan x$	$\frac{1}{1+x^2}$	$\operatorname{arccot} x$	$\frac{-1}{1+x^2}$
$\sinh x$	$\cosh x$	$\cosh x$	$\sinh x$

Geometry Crib Sheet

Pythagorean Identity $a^2+b^2=c^2$

Circle: radius r	$A=\pi r^2$	$c=2\pi r$
Box: dimensions x, y, z	$V=xyz$	$A=2(yz+xz+xy)$
Sphere: radius r	$V=\frac{4}{3}\pi r^3$	$A=4\pi r^2$
Right pyramid: height h dim x, y	$V=\frac{1}{3}hxy$	$A=xy+x\sqrt{(y/2)^2+h^2}+y\sqrt{(x/2)^2+h^2}$
Cylinder: height h radius r	$V=\pi hr^2$	$A=2\pi r(h+r)$
Right Cone: height h radius r	$V=\frac{\pi}{3}hr^2$	$A=\pi r\left(r+\sqrt{r^2+h^2}\right)$
Torus: radii $R>r$	$V=2\pi^2r^2R$	$A=4\pi^2rR$
Tetrahedron: edge x	$V=\frac{1}{6\sqrt{2}}x^3$	$A=\sqrt{3}x^2$
Octahedron: edge x	$V=\frac{\sqrt{2}}{3}x^3$	$A=2\sqrt{3}x^2$
Dodecahedron: edge x	$V=\frac{15+7\sqrt{5}}{4}x^3$	$A=3\sqrt{20+10\sqrt{5}}x^2$
Icosahedron: edge x	$V=\frac{5\left(3+\sqrt{5}\right)}{12}x^3$	$A=5\sqrt{3}x^2$

1. (a) (3 points) State the Mean Value Theorem.

   Solution: Suppose that $f : [a, b] \to \mathbb{R}$ is differentiable on interval (a, b). Then there is some point $c \in (a, b)$ such that $f'(c) = \frac{f(b)-f(a)}{b-a}$

   (b) (6 points) Suppose that $f : \mathbb{R} \to \mathbb{R}$ is a real twice differentiable real function defined on all of $\mathbb{R}$ with $f(0) = 2$ and $f'(0) = 5$ and $f(1) = 5$ and $f'(1) = 7$. Which of the following MUST true? Circle all that apply.

   A. f has a local maximum at $x = 1$.
   B. f does not have a zero in $[0, 1]$
   C. f' must achieve the value 3 somewhere in $[0, 1]$
   D. f must be concave down somewhere in $[0, 1]$
   E. f must be concave up somewhere in $[0, 1]$

   Solution: (A) is false or $f'(1)$ would be 0. (B) could be false or true, we do not have enough information. (C) is true by the Mean Value Theorem. (D) is true as f' decreased from 5 to 3. (E) is true as f' increased from 5 to 7

2. (9 points) Let g be the real function defined by

$$g(x) = x \ln |x|$$

Find the critical points, where g is increasing, and where g is decreasing.

The critical point(s) of g is (are) ____________________

g is increasing on ____________________

g is decreasing on ____________________

Solution: Observe that $g'(x) = \ln |x| + 1$. which does not exist for $x = 0$ and is equal to 0 for $x = \pm\frac{1}{e}$. So the critical points of g are $0, \pm\frac{1}{e}$. Checking the sign we see that g is increasing on $(-\infty, -1/e) \cup (1/e, \infty)$ and g is decreasing on $(-1/e, 0) \cup (0, 1/e)$.

3. (12 points) Let $f(x) = \frac{1}{3}x^3 - x^2 - 3x + 10$ be defined on the domain $[-5, 5]$. Compute the absolute maximum, minimum and the x-values at which they occur.

The absolute maximum is ________________ which occurs at $x =$________________.

The absolute minimum is ________________ which occurs at $x =$________________.

Solution: Compute $f'(x) = x^2 - 2x - 3 = (x-3)(x+1)$. Take the zeros and the endpoint to see the critical points of f are $-1, 3, -5, 5$. Considering the sign of f' we see that -1 and 5 are local maxima and -5 and 3 are local minima. We evaluate to determine the global extrema. Compute $f(-5) = \frac{-125}{3}$ and $f(3) = 1$ and $f(-1) = f(5) = \frac{35}{3}$. So the global maximum is $\frac{35}{3}$ achieved at -1 and 5. The global minimum is $\frac{-125}{3}$ achieved at -5.

4. (9 points) Find all the critical points of the function $h(x) = |x-1| - 2|x+3| - 4|x-5|$. Decide if each point is a local minimum, local maximum, or neither.

   The critical points of h are ________________.

   Local maxima occur at ________________

   Local minima occur at ________________

   Non-extremal critical points occur at ________________

   Solution: Compute the derivative of each term in the sum separately

$$h'(x) = \begin{cases} 1 & \text{if } x > 1 \\ -1 & \text{if } x < 1 \end{cases} + \begin{cases} -2 & \text{if } x > -3 \\ 2 & \text{if } x < -3 \end{cases} + \begin{cases} -4 & \text{if } x > 5 \\ 4 & \text{if } x < 5 \end{cases}$$

so the derivative is never 0, but h has three critical points where the derivative does not exist at $-3, 1$, and 5.

$$\xleftarrow{h'=5} -3 \xrightarrow{1} 1 \xrightarrow{3} 5 \xrightarrow{-5}$$

Using the first derivative test we see that $x = -3$ and $x = 1$ are not extrema. But 5 a local maximum. In fact it is the global maximum.

5. (9 points) Let $g : \mathbb{R} \to \mathbb{R}$ be the differentiable real function defined by

$$g(x) = \frac{1}{20}x^5 - \frac{1}{6}x^4 + \frac{1}{6}x^3 + x - 12$$

Find the inflection points, where g is concave up, and where g is concave down.

The inflection point(s) of g is (are) ____________________

g is concave up on ____________________

g is concave down on ____________________

Solution: Compute $g''(x) = x^3 - 2x^2 + x = x(x-1)^2$ so the possible inflection points are 0 and 1. Then

$$g'' : \xleftarrow{-} 0 \xrightarrow{+} 1 \xrightarrow{+}$$

So g is concave up on $(0,1) \cup (1,\infty)$ and g is concave down on $(-\infty, 0)$. The concavity changes at $x = 0$ so that is the only inflection point.

BONUS (5 points): Let f be a twice differentiable real function. Suppose that f achieves its maximum value of 1 at every integer in $[1, 100]$. What is the smallest possible number of zeros that f' could have and how do you know?

Solution: 199. $f'(x) = 0$ if x is an integer from 1 to 100. By Rolle's Theorem $f'(c) = 0$ for some c between every pair of consecutive integers from 1 to 100. That guarantees $100 + 99$ zeros. A normalized sum of gaussian functions, e.g. $f(x) = \sum_{n=1}^{100} a_n e^{-(x-n)^2}$ can achieve this, so there need not be more