Math 1551-G
Fall 2015
Exam 2
9 October 2015
Time Limit: 50 Minutes

Name: Solutions

This exam contains 7 pages (including this cover page) and 6 questions. There are 48 points in total. Write explanations clearly and in complete thoughts. No calculators or notes may be used. Put your name on every page.

Grade Table

Question	Points	Score
1	8	
2	12	
3	9	
4	6	
5	8	
6	5	
Total:	48	

Formal Symbols Crib Sheet

$f : A \to B$	function with domain A & codomain B	$\mathbb{N}$	natural numbers
$f \circ g$	composition of functions	$\mathbb{Z}$	integers
f^{-1}	inverse function	$\mathbb{Q}$	rational numbers
$\lim_{x\to a}$	limit as x approaches a	$\mathbb{R}$	real numbers
$\lim_{x\to a^-}$	limit from below	(a, b)	open interval a to b
$\lim_{x\to a^+}$	limit from above	$[a, b]$	closed interval a to b
$\subset$	subset of	$\in$	element of
$\cap$	intersection	$\cup$	union
$\mapsto$	maps to	f'	derivative
$\frac{d}{dx}$	derivative with respect to x		

1. (a) (3 points) Give the definition of the derivative of a real function f.

The derivative of f is the function f' defined by

$$f'(x) = \lim_{\Delta x \to 0} \frac{f(x+\Delta x) - f(x)}{\Delta x}$$

(b) (5 points) Which of the following is equal to the derivative $f'(a)$ of f at $a \in \mathbb{R}$? Circle ALL that apply.

A. the slope of the line tangent to the graph of f at the point $(a, f(a))$

B. the angle between the x-axis and the line from $(0, 0)$ to $(a, f(a))$

C. the instantaneous rate of change of f at a

D. the negative reciprocal of the slope of the line normal to the graph of f at the point $(a, f(a))$

E. the angle between the x-axis and the line tangent to the graph of f at the point $(a, f(a))$

2. The table below contains some values of real functions f and g and their derivatives f' and g' evaluated at different values of x.

x	$f(x)$	$f'(x)$	$g(x)$	$g'(x)$
0	1	2	3	4
1	5	6	7	8
2	9	10	11	12
3	13	14	15	16

(a) (3 points) What is the derivative of the sum $f+g$ at 0?

$(f+g)'(0) =$ $2+4=6$

(b) (3 points) What is the derivative of the product fg at 0?

$(fg)'(0) =$ $1\cdot 4 + 2\cdot 3 = 10$

(c) (3 points) What is the derivative of the composition $f \circ g$ at 0?

$(f\circ g)'(0) =$ $f'(g(0))\cdot g'(0) = 14\cdot 4 = 56$

(d) (3 points) Let $h(x) = f(2x)$. What is the derivative of h at 1?

$h'(1) =$ $2f'(2) = 2\cdot 10 = 20$

3. (a) (3 points) Compute the derivative

$$\frac{d}{dx}\left(\sin(x)\sin(x)\sin(x)\right) =$$

$$\frac{d}{dx} \sin^3(x) = 3 \cdot \sin^2(x) \cdot \cos(x)$$

(b) (3 points) Compute the derivative

$$\frac{d}{dx}\left(\sin\left(\sin\left(\sin(x)\right)\right)\right) =$$

$$\cos(\sin(\sin x)) \cdot \cos(\sin x) \cdot \cos(x)$$

(c) (3 points) Compute the derivative

$$\frac{d}{dx}\left(\frac{x^5 5^x}{\log_5 x}\right) =$$

$$\frac{\left(5x^4 \cdot 5^x + x^5 5^x \ln 5\right)\log_5 x + x^{4} 5^x \cdot \frac{1}{\ln 5}}{\left(\log_5 x\right)^2}$$

4. (6 points) Consider $h(x) = (5x - \frac{1}{3x})^2$. Find the third derivative $h'''(x)$. (Hint: simplify before taking additional derivatives.)

$h'''(x) =$

$$h'(x) = 2\left(5x - \frac{1}{3x}\right)\left(5 + \frac{1}{3x^2}\right)$$

$$= 50x - \frac{2}{9x^3}$$

$$h''(x) = 50 + \frac{2}{3x^4}$$

$$h'''(x) = \frac{-8}{3x^5}$$

5. (8 points) Consider the piecewise defined real function g

$$g(x) = \begin{cases} 4+2x & \text{if } x < -3 \\ 2+2x & \text{if } -3 \leq x \leq -1 \\ \frac{4}{1+x^2} & \text{if } -1 < x < 1 \\ |2-x|+1 & \text{if } x \geq 1 \end{cases}$$

What is the domain of g'? Compute a formula for the derivative g' at all points where it is defined.

The domain of g' is $(-\infty,-3) \cup (-3,-1) \cup (-1,1) \cup (1,2) \cup (2,\infty)$

ie, all real numbers except for $-3, -1, 1,$ and 2

$$g'(x) = \begin{cases} 2 & \text{if } x \in (-\infty,-3) \cup (-3,-1) \\ \frac{-8x}{(1+x^2)^2} & \text{if } x \in (-1,1) \\ -1 & \text{if } x \in (1,2) \\ 1 & \text{if } x \in (2,\infty) \end{cases}$$

- There are jump discontinuities at $-3, -1$ so g is not differentiable there.

There are sharp points for g ($\Leftrightarrow$ jump discontinuities for g') at 1 and 2.

6. (5 points) Compute the limit. (Hint: This limit is a derivative of some function at a point. Compute the derivative and use it to evaluate the limit.)

$$\lim_{x\to 0} \frac{e^{3(1+x)^2} - e^3}{x} = \frac{d}{du} e^{3u^2}\Big|_{u=1}$$

$$= 6u\, e^{3u^2}\Big|_{u=1}$$

$$= 6e^3$$

BONUS: (5 points) Explain how you can derive the formula for $\frac{d}{dx}\arctan x$ as a rational function in x from the chain rule and the fact that $\frac{d}{dx}\tan x = \sec^2 x$.

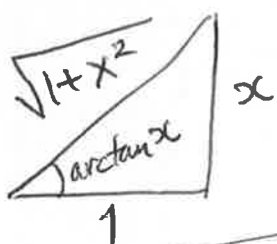
As tan and arctan are inverse functions:

$\tan(\arctan(x)) = x$.

So $\sec^2(\arctan(x))\, \frac{d}{dx}\arctan x = 1$ by the chain rule.

$$\Longrightarrow \frac{d}{dx}\arctan x = \frac{1}{\sec^2(\arctan(x))}.$$

To simplify this as a ration function observe that we have the triangle:



so $\sec(\arctan(x)) = \sqrt{1+x^2}$

Then $\boxed{\frac{d}{dx}\arctan x = \frac{1}{1+x^2}}$