Wavelet-Based Compression of Signals

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Hilbert Spaces

Signals

Signals are vectors in the Hilbert space

$$L_2 = \left\{ v : \mathbb{R} \to \mathbb{C} \,\middle|\, \int_{\mathbb{R}} |v(t)|^2 dt \right\}$$

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Digital Signals

In practice we typically use *digital signals* in the Hilbert space

$$\ell_2\mathbb{Z}_N = \{z : \mathbb{Z}_N \to \mathbb{C}\}$$

Hilbert Spaces

Inner Product

Hilbert spaces are equipped with the inner product $\langle \cdot | \cdot \rangle : H \times H \to \mathbb{C}$. For L_2

$$\langle u|v\rangle = \int_{\mathbb{R}} \overline{u(t)}v(t)dt$$

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and are Cauchy complete with respect the the metric/norm

$$\|u - v\| = \langle u - v|u - v\rangle^{\frac{1}{2}}$$

Bases

If $\{a_k | k \in \mathbb{Z}\}$ is a complete othonormal basis for a Hilbert space H then any $v \in H$ can be written in the form

$$v = \sum_{k \in \mathbb{Z}} \langle a_k | v \rangle a_k$$

Fourier Transform

If $f \in L_2$ we call its Fourier transform $\hat{f}$ the function

$$\hat{f}(\omega) = \frac{1}{\sqrt{2\pi}} \int_{\mathbb{R}} f(t) e^{-it\omega} dx$$

(when it exists).

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That is (almost)

$$f(t) = \int_{\mathbb{R}} \langle \frac{e^{it\omega}}{\sqrt{2\pi}} | f \rangle \frac{e^{it\omega}}{\sqrt{2\pi}} d\omega$$

Fourier Transform

If $z \in \ell_2\mathbb{Z}_N$ we call its Fourier transform $\hat{z}$

$$\hat{z}_m = \sum_{n=0}^{N-1} z_n \frac{e^{-i2\pi mn/N}}{\sqrt{N}}$$

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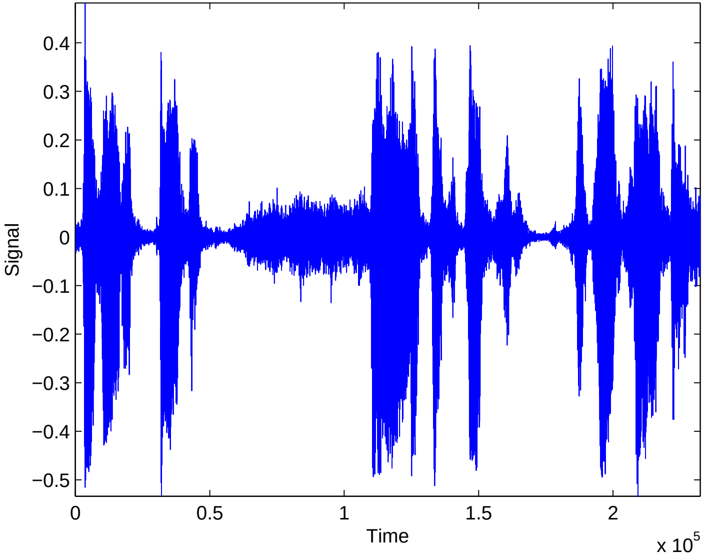
We have

$$z_n = \sum_{m=0}^{N-1} \langle \frac{e^{i2\pi mn/N}}{\sqrt{N}} | z \rangle \frac{e^{i2\pi mn/N}}{\sqrt{N}}$$

The Walken Signal z



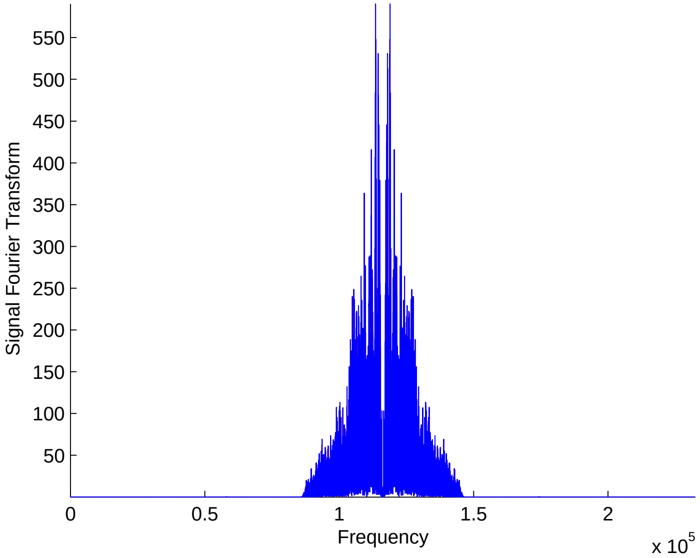
Play

Fourier Compression

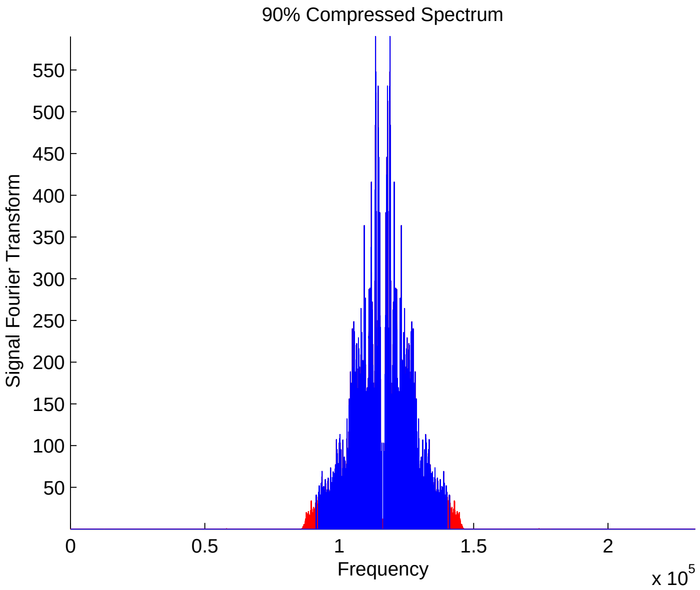
An easy low-loss compression

Discard the lowest p^{th} percent of Fourier coefficients, replacing them by zero.

$\hat{z}$ The Walken Spectrum (Fourier Transform)



$\hat{z}$ The Walken Spectrum 90% Compressed



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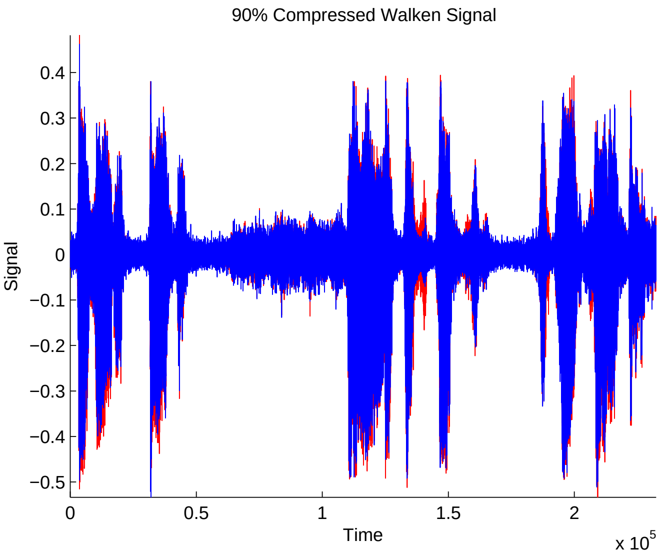

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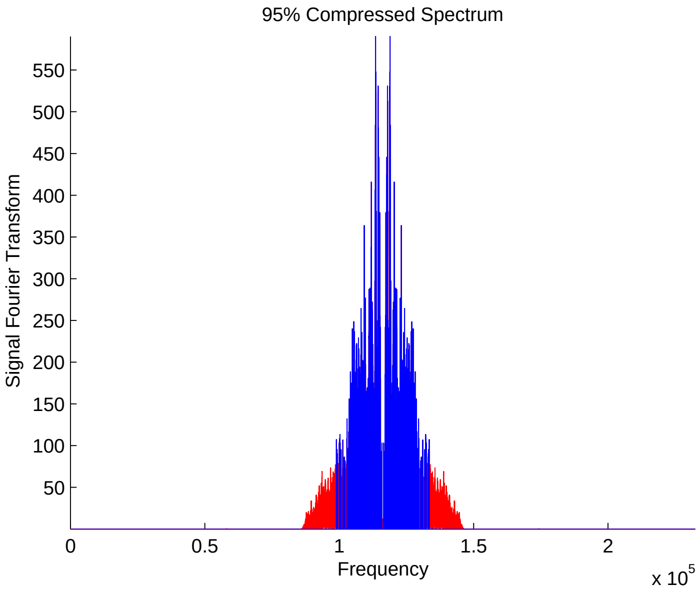

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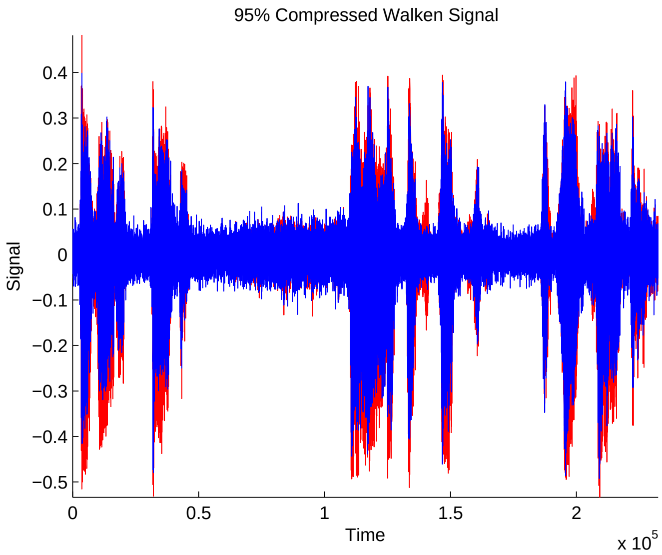

Play

Problems

Practial Objection

$$z_n = \sum_{m=0}^{N-1} \hat{z}_m \frac{e^{i2\pi mn/N}}{\sqrt{N}}$$

$$f(t) = \int_{-\infty}^{\infty} \hat{f}(\omega) \frac{e^{it\omega}}{\sqrt{2\pi}} d\omega$$

Delocalization

1. Signal is delocalized in *frequency* space
2. Spectrum is delocalized in *temporal* space

Localization in Streaming

Play

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Delocalization

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Impractical Objection

$$\frac{e^{it\omega}}{\sqrt{2\pi}} \notin L_2$$

This isn't a basis for L_2.

Wavelets

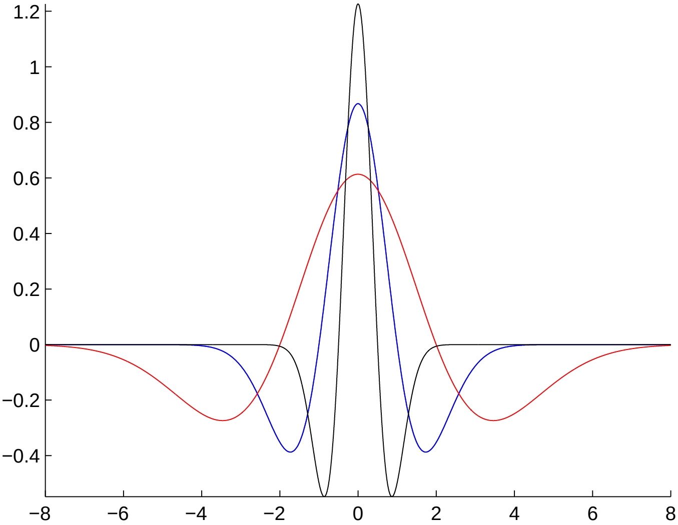
Wavelets in L_2

A wavelet in L_2 is $\psi \in L_2$ such that

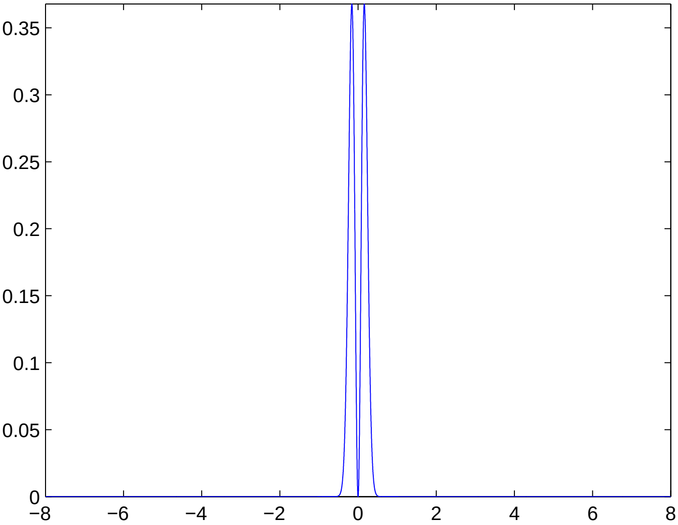
$$\psi_{j,k}(t) = 2^{j/2}\psi(2^j t - k)$$

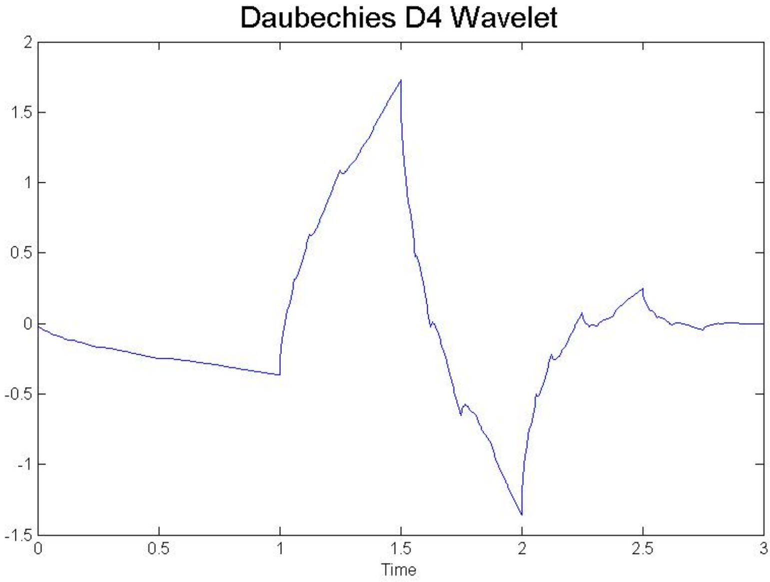
satisfy $\{\psi_{j,k} \mid j, k \in \mathbb{Z}\}$ is a complete, orthonormal basis.

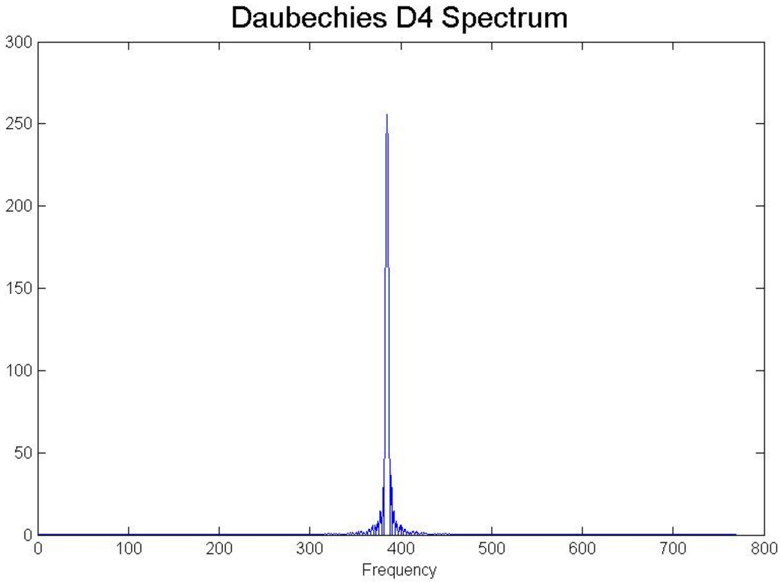
Mexican Hat Wavelet



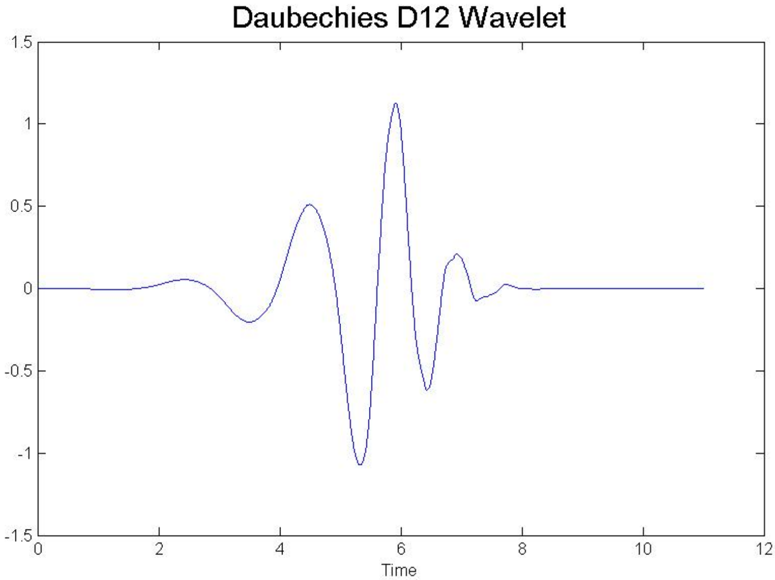
Mexican Hat Fourier Transform

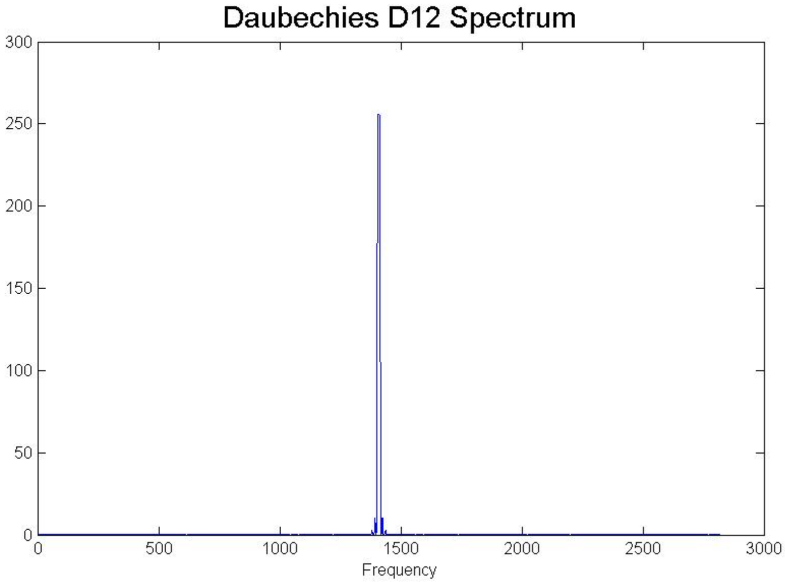



Daubechies D4 Wavelet
2
1.5
1
0.5
0
-0.5
-1
-1.5
0
0.5
1
1.5
2
2.5
3
Time



Daubechies D4 Spectrum
300
250
200
150
100
50
0
0
100
200
300
400
500
600
700
800
Frequency


Daubechies D12 Wavelet
1.5
1
0.5
0
-0.5
-1
-1.5
0
2
4
6
8
10
12
Time


Daubechies D12 Spectrum
300
250
200
150
100
50
0
0
500
1000
1500
2000
2500
3000
Frequency

k^{th} Translation

If $z \in \ell_2\mathbb{Z}_N$

$$R_k z_n = z_{n-k}$$

Wavelets in $\ell_2\mathbb{Z}_N$

A first stage wavelet pair is a pair of vectos $u, v \in \ell_2\mathbb{Z}_N$ such that

$$B = \{R_{2k}u \mid k = 0, \ldots, N/2 - 1\} \cup \{R_{2k}v \mid k = 0, \ldots, N/2 - 1\}$$

is a complete othonormal basis fo $\ell_2\mathbb{Z}_N$

Low Pass and High Pass

A wavelet pair u, v must satisfy

$$|\hat{u}(n)|^2 + |\hat{u}(n + N/2)|^2 = 2$$

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z expansion

$$z = \sum_{n=0}^{N/2-1} \langle R_{2k}u|z\rangle R_{2k}u + \sum_{n=0}^{N/2-1} \langle R_{2k}v|z\rangle R_{2k}v$$

Wavelets

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1. The first term contains an approximation.

Wavelets

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z expansion

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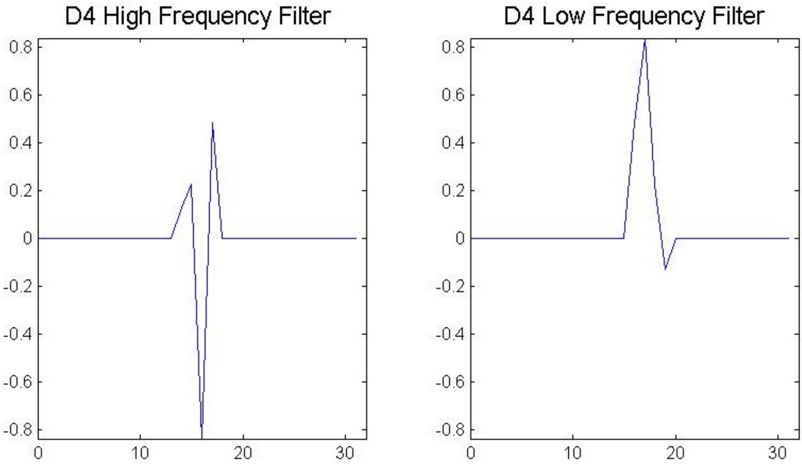
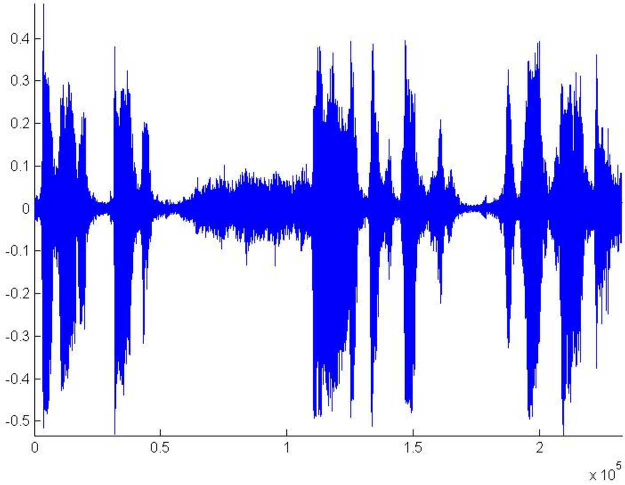
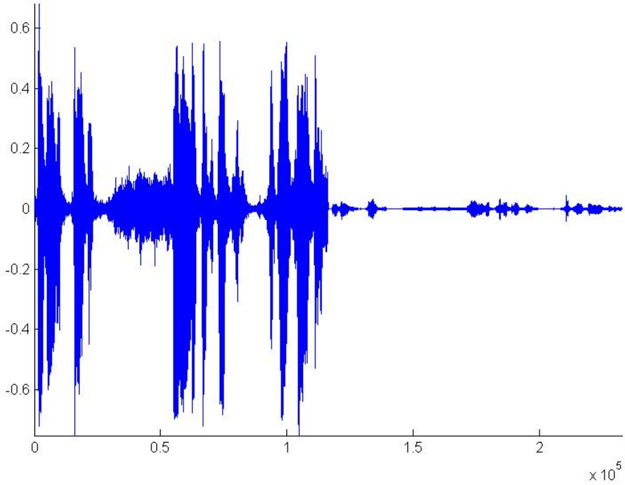
1. The first term contains an approximation.
2. The second term contains the details.


D4 High Frequency Filter
D4 Low Frequency Filter
0.8
0.6
0.4
0.2
0
-0.2
-0.4
-0.6
-0.8
0
10
20
30

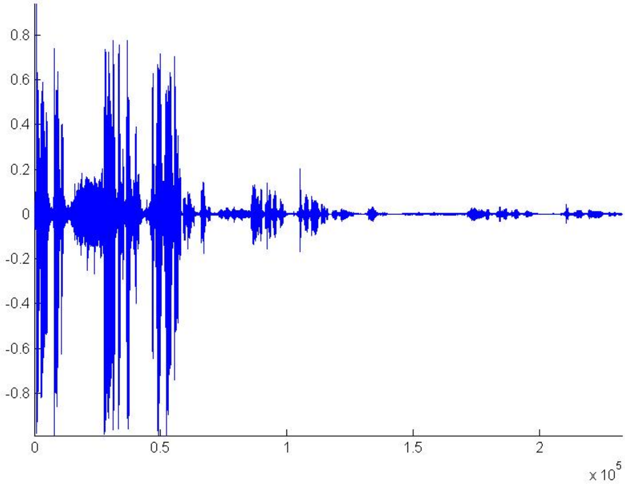
The Walken Signal



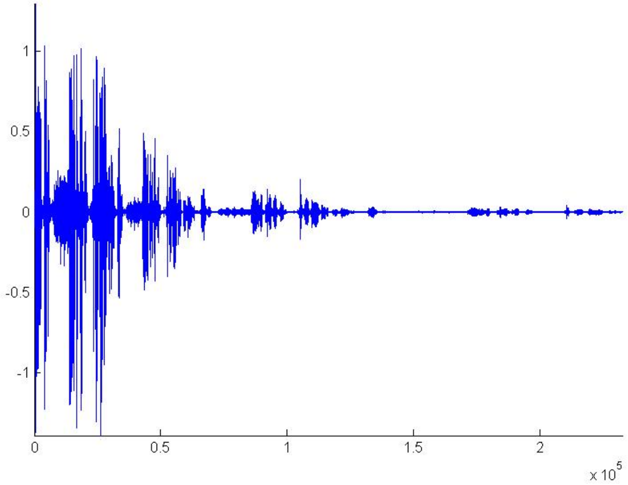
1st Stage Wavelet Representation



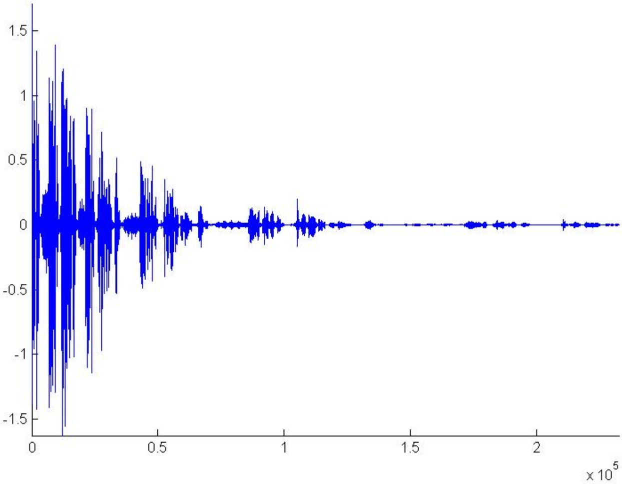
2nd Stage Wavelet Representation



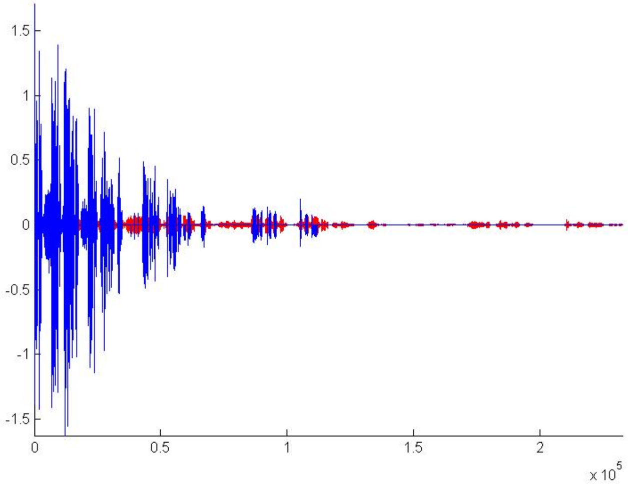
3rd Stage Wavelet Representation



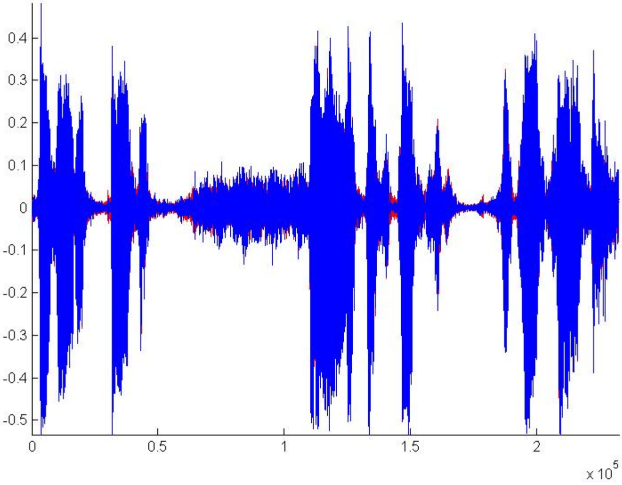
4th Stage Wavelet Representation



4th Stage Wavelet Representation 90% Compression

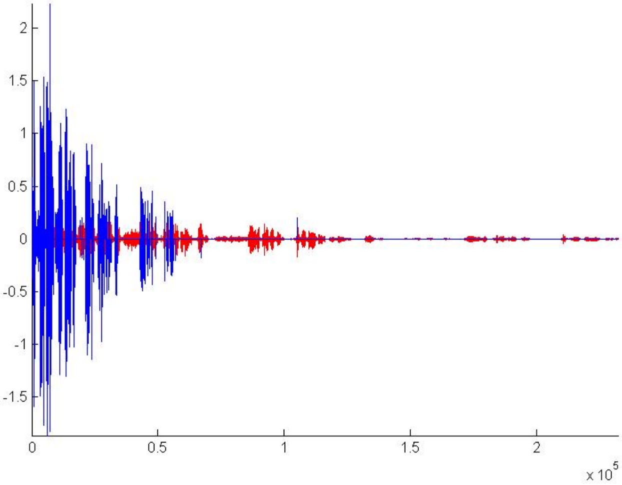


Reconstructed 90% Compression

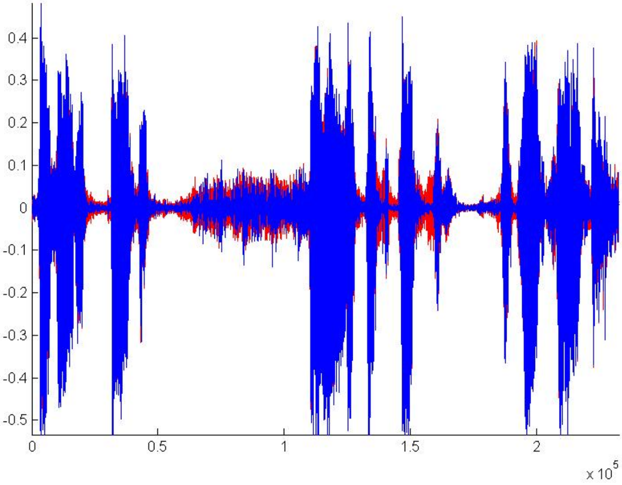


Play

5th Stage Wavelet Representation 95% Compression



Reconstructed 95% Compression



Play

Sound Comparison

Original

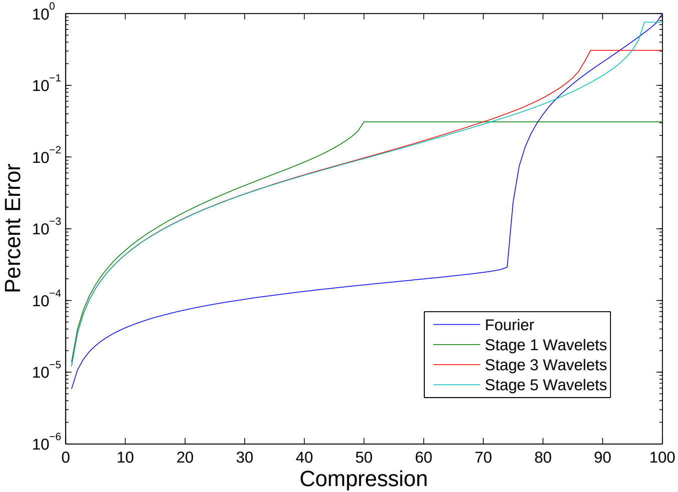
.9 Compressed Fourier

.95 Compressed Fourier

.9 Compressed Wavelet

.95 Compressed Wavelet

Percent Error Comparisons



Acknowledgments

Thank You

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